

## 4.2.4 oporachten

1. a) JA want

- $(1, 1, 0) \in U_1$  dus  $U_1 \neq \emptyset$
- $(x, y, 0) \cdot \lambda + (a, b, 0) \cdot \mu$   
 $= (\lambda x + \mu a, \lambda y + \mu b, 0) \in U_1$

b) NEEN want

$$\bullet (2, 3, 4) \cdot \pi + (0, 0, 0) \cdot 2 = (2\pi, 3\pi, 4\pi) \notin U_2$$

c) NEEN want

$$\bullet (5, 6, 7) \cdot (-3) + (0, 0, 0) \cdot 2 = (-15, -18, -21) \notin U_3$$

d) NEEN want

$$\bullet (-1, 2, 3) \cdot 0 + (-2, 3, 7) \cdot 0 = (0, 0, 0) \notin U_4$$

e) JA want

- $(2, 1, 0) \in U_5$  dus  $U_5 \neq \emptyset$
- $(2y_1, y_1, z_1) \cdot \lambda + (2y_2, y_2, z_2) \cdot \mu$   
 $= (2\lambda y_1 + 2\mu y_2, \lambda y_1 + \mu y_2, \lambda z_1 + \mu z_2) \in U_5$

f) JA want

- $(2, 1, 0) \in U_6$  dus  $U_6 \neq \emptyset$
- $(2y_1 + 3z_1, y_1, z_1) \cdot \lambda + (2y_2 + 3z_2, y_2, z_2) \cdot \mu$   
 $= (2\lambda y_1 + 3\lambda z_1 + 2\mu y_2 + 3\mu z_2, \lambda y_1 + \mu y_2, \lambda z_1 + \mu z_2)$   
 $\in U_6$  want

$$\begin{aligned} & -2\lambda y_1 - 3\lambda z_1 - 2\mu y_2 - 3\mu z_2 + 2\lambda y_1 + 2\mu y_2 + 3\lambda z_1 + 3\mu z_2 \\ & = 0 \end{aligned}$$

g) JA want

- $(1, 0, -1) \in U_7$  dus  $U_7 \neq \emptyset$
- $(-y_1 - z_1, y_1, z_1) \lambda + \mu (-y_2 - z_2, y_2, z_2)$   
 $= (-\lambda y_1 - \lambda z_1 - \mu z_2 - \mu y_2, \lambda y_1 + \mu y_2, \lambda z_1 + \mu z_2) \in U_7$

h) NEEN want

$$(1, 2, 2) \cdot 4 + (0, 0, 0) \cdot 7 = (4, 8, 8) \notin U_8$$

2. a) JA want

- $f: \mathbb{R} \rightarrow \mathbb{R}: x \rightarrow 0 \in U_1$  dus  $U_1 \neq \emptyset$
- Kies  $f_1, f_2$  willekeurig zo dat  $f_1(5) = 0$  en  $f_2(5) = 0$

$$\text{TB: } (\lambda f_1 + \mu f_2)(5) = 0$$

$$\begin{aligned} \text{Bewijs: } (\lambda f_1 + \mu f_2)(5) &= (\lambda f_1)(5) + (\mu f_2)(5) \\ &= \lambda f_1(5) + \mu f_2(5) \\ &= \lambda \cdot 0 + \mu \cdot 0 \\ &= 0 \end{aligned}$$

b) JA want

- $f: \mathbb{R} \rightarrow \mathbb{R}: x \rightarrow \sin x \in U_2$  dus  $U_2 \neq \emptyset$
- Kies  $f_1, f_2$  willekeurig met  $f_1, f_2 \in U_2$

Dan kunnen we een  $M \in \mathbb{R}$  vinden zo dat voor alle  $x \in \mathbb{R}$  geldt  $|f_1(x)| < M$  en  $|f_2(x)| < M$

$$\begin{aligned} \text{Nu, } |\lambda f_1 + \mu f_2| &\leq |\lambda| |f_1| + |\mu| |f_2| \\ &\leq |\lambda| M + |\mu| M. \end{aligned}$$

Dus nog  $\lambda f_1 + \mu f_2$  is begrensd.

c) JA want

- $f: \mathbb{R} \rightarrow \mathbb{R}: x \rightarrow 4 \in U_3$  dus  $U_3 \neq \emptyset$
  - Kies  $f_1, f_2$  willekeurig met  $f_1, f_2 \in U_3$
- $$\begin{aligned} (\lambda f_1 + \mu f_2)(0) &= \lambda f_1(0) + \mu f_2(0) \\ &= \lambda f_1(1) + \mu f_2(1) = (\lambda f_1 + \mu f_2)(1) \end{aligned}$$

d) NEEN want

$$\bullet 2 \cdot f_1 + 5 f_2 \notin U_4$$

$$\text{met } f_1: \mathbb{R} \rightarrow \mathbb{R}: x \rightarrow 4$$

$$f_2: \mathbb{R} \rightarrow \mathbb{R}: x \rightarrow 0$$

e) JA want

$$\bullet f: \mathbb{R} \rightarrow \mathbb{R}: x \rightarrow x^2 \in U_5 \text{ dus } U_5 \neq \emptyset$$

\bullet Kies een willekeurige  $f_1, f_2 \in U_5$

$$(\lambda f_1 + \mu f_2)(x) = \lambda f_1(x) + \mu f_2(x)$$

$$= \lambda f_1(-x) + \mu f_2(-x)$$

$$= (\lambda f_1 + \mu f_2)(-x)$$

f) JA want

$$\bullet f: \mathbb{R} \rightarrow \mathbb{R}: x \rightarrow 0 \in U_6 \text{ dus } U_6 \neq \emptyset$$

\bullet Kies een willekeurige  $f_1, f_2 \in U_6$

$$(\lambda f_1 + \mu f_2)'' = \lambda f_1'' + \mu f_2''$$

Omdat  $f_1''$  en  $f_2''$  continu zijn zal  $\lambda f_1'' + \mu f_2''$  ook continu zijn.

(functies opgebouwd uit continue functies, zijn zelf ook continu)

3. a) NEEN want

$$(-7)(x_n)_{n \in \mathbb{N}} + (-5) \cdot (y_n)_{n \in \mathbb{N}} \notin U_1$$

$$\text{met } (x_n)_{n \in \mathbb{N}} = n$$

$$(y_n)_{n \in \mathbb{N}} = n^2$$

b) NEEN want

$$0 \cdot (x_n)_{n \in \mathbb{N}} + 0 \cdot (y_n)_{n \in \mathbb{N}} \notin U_2$$

$$\text{met } (x_n)_{n \in \mathbb{N}} = -n$$

$$(y_n)_{n \in \mathbb{N}} = n$$

c) JA want

- $(x_n)_{n \in \mathbb{N}} = 0 \in U_3$  aus  $U_3 \neq \emptyset$
- Kies twee willekeurige rijen  $(x_n)_{n \in \mathbb{N}}$  en  $(y_n)_{n \in \mathbb{N}} \in U_3$ . Dan kunnen we een  $n_1 \in \mathbb{N}$  vinden zo dat voor alle  $n \geq n_1$  geldt  $x_n = 0$ . En we kunnen dan een  $n_2 \in \mathbb{N}$  vinden zo dat voor alle  $n \geq n_2$  geldt  $y_n = 0$ . Neem nu  $n_0 = \max\{n_1, n_2\}$ . Dan geldt voor alle  $n \geq n_0$  dat  $n \geq n_1$  en  $n \geq n_2$ . Bovendien geldt voor alle  $n \geq n_0$ :  $\lambda(x_n)_{n \in \mathbb{N}} + \mu(y_n)_{n \in \mathbb{N}} = \lambda \cdot 0 + \mu \cdot 0 = 0$

4. a) We weten dat  $\forall y_1, y_2 \in Y. \lambda y_1 + \mu y_2 \in Y$   
en  $\forall x_1, x_2 \in X. \lambda x_1 + \mu x_2 \in X$   
Nu,  $\lambda(x_1 + y_1) + \mu(x_2 + y_2)$   
 $= (\lambda x_1 + \mu x_2) + (\lambda y_1 + \mu y_2) \in X + Y$

b) Bewijs uit het ongerijmde:

Neem  $x_1, x_2 \in X \cap Y$  (\*)

Veronderstel nu dat  $\exists \lambda, \mu \in \mathbb{R}$  zo dat  $\lambda x_1 + \mu x_2 \notin X \cap Y$ . Dat betekent dat  $\lambda x_1 + \mu x_2$  geen element is van  $X$  of  $Y$ .

Maar omdat  $X$  en  $Y$  deelruimten zijn, kan ofwel  $x_1$  ofwel  $x_2$  niet in  $X$  of  $Y$  zitten. Dit is tegenspraak met (\*).

5. De deelruimte is van volgende vorm

$$\lambda \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \\ 4 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ -3 \\ 5 \\ 2 \end{pmatrix} \quad \forall \lambda, \mu, \gamma \in \mathbb{R}$$

We kijken dus of er een  $\lambda, \mu, \gamma$  bestaat zo dat

$$\left\{ \begin{array}{l} \lambda + 2\mu = 5 \\ -\lambda + \mu - 3\gamma = 1 \\ 2\lambda - \mu + 5\gamma = 0 \\ 3\lambda + 4\mu + 2\gamma = 11 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \lambda = 5 - 2\mu \\ -5 + 2\mu + \mu - 3\gamma = 1 \\ 2\lambda - \mu + 5\gamma = 0 \\ 3\lambda + 4\mu + 2\gamma = 11 \end{array} \right.$$

$$\left\{ \begin{array}{l} \lambda = 5 - 2\mu \\ \gamma = \mu - 2 \\ 10 - 4\mu - \mu + 5\mu - 10 = 0 \\ 15 - 6\mu + 4\mu + 2\mu - 4 = 11 \end{array} \right. \mu = 10 \Leftrightarrow \left\{ \begin{array}{l} \lambda = -15 \\ \gamma = 8 \\ 0 = 0 \\ 0 = 0 \end{array} \right.$$

→ het is een deel van de ruimte

$$\left\{ \begin{array}{l} \lambda + 2\mu = 3x \\ -\lambda + \mu - 3\gamma = 3 \\ 2\lambda - \mu + 5\gamma = x - 5 \\ 3\lambda + 4\mu + 2\gamma = 7x - 2 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \lambda = 3x - 2\mu \\ -3x + 2\mu + \mu - 3\gamma = 3 \\ 2\lambda - \mu + 5\gamma = x - 5 \\ 3\lambda + 4\mu + 2\gamma = 7x - 2 \end{array} \right.$$

$$\left\{ \begin{array}{l} \lambda = 3x - 2\mu \\ \gamma = -1 - x + \mu \\ 6x - 4\mu - \mu - 5 - 5x + 5\mu = x - 5 \\ 9x - 6\mu + 4\mu - 2 - 2x + 2\mu = 7x - 2 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} 0 = 0 \\ 0 = 0 \end{array} \right.$$

→ voor elke  $x$  is er een op, voor het stelsel

$$6. \left\{ \begin{array}{l} \lambda + 2\mu = 1 \\ -3\lambda - \mu = k \\ 2\lambda + \mu = 5 \end{array} \right.$$

$$\left\{ \begin{array}{l} \lambda = 3 \\ -9 + 1 = k \\ \mu = -1 \end{array} \right.$$

$$\left\{ \begin{array}{l} \lambda = 1 - 2\mu \\ -3\lambda - \mu = k \\ 2 - 4\mu + \mu = 5 \end{array} \right.$$

$$\left\{ \begin{array}{l} k = -8 \end{array} \right.$$

$$7 \text{ stel } \begin{cases} a + 4b = 2c + d \\ 2a + 5b = c + d \\ 3a + 6b = d \end{cases} \quad \begin{cases} a + 4b = -2a - 2b + 3a + 6b \\ c = -a - b \\ d = 3a + 6b \end{cases}$$

$$\begin{cases} 0 = 0 \\ c = -a - b \\ d = 3a + 6b \end{cases}$$

→ voor willekeurige  $a, b$  bestaan er  $c, d$  zo dat

$$a \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + b \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = c \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

8. Kies een willekeurige  $n \in \mathbb{N}$

dan is  $\lambda_0 e_0 + \lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_n e_n =$   
 $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n, 0, 0, 0, \dots)$

→ ze brengen dus de deelruimte voort  
 via rijen met een staart van nullen

9. Beschouw het stelsel in matrixvorm

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & \dots & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \quad (*)$$

Dit stelsel is oplosbaar als en slechts als er een  $x$  bestaat zo dat  $(*)$  geldt. Schrijven we  $(*)$  nu uit volgens de regels van het product van matrices, bekommen we:

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

of nog

$$x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

of nog

$$x_1 A_1 + x_2 A_2 + x_3 A_3 + \dots + x_n A_n = B$$

of nog

$$B \in \text{Vect} \{A_1, A_2, \dots, A_n\}$$

10. De oplossingsruimte van stelsel is

$$O = \{(-2\mu - 2\lambda, \mu + 3\lambda, \mu, -2\lambda, \lambda)\}$$

een voortbrengend deel is dan bv.

$$\{(-2, 1, 1, 0, 0), (-2, 3, 0, -2, 1)\}$$

## Extra oefeningen opdracht 2

extra oef 1.

1.  $\{M_1, M_2, M_3, M_4\}, \{M_1, M_2, M_3\}, \{\pi_1, \pi_2\}$

2.  $\{M_1, M_2\}, \{M_1\}, \emptyset$

3.  $\{M_1, M_2\}, \{M_2, M_3\}, \{\pi_4, \pi_1\}$

4. 2

extra oef 2.

Stel  $x = y = t = 0$

$\rightarrow \{0, 3, 6\}$

Stel  $x = y = t = 1$

$\rightarrow \{2, 2, -4\}$

Stel  $x = y = t = -1$

$\rightarrow \{-2, 2, 12\}$

$$\begin{cases} 3\lambda_2 + 6\lambda_3 = 0 \\ 2\lambda_1 + 2\lambda_2 - 4\lambda_3 = 0 \\ -2\lambda_1 + 2\lambda_2 + 12\lambda_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda_2 = -2\lambda_3 \\ \lambda_1 = 4\lambda_3 \\ 0 = 0 \end{cases}$$

$\rightarrow$  we DENKEN lineair afhankelijk

$h = 2g - 4f$

$\rightarrow$  NIET vrij

extra oef 3.

Stel  $x = 0$

$\rightarrow \{1, 0, 1\}$

afleiden +  $x = 0$

$\rightarrow \{2, 1, 4\}$

nog eens afleiden +  $x = 0$

$\rightarrow \{4, 2, 16\}$

$$\begin{cases} \lambda_1 + \lambda_3 = 0 \\ 2\lambda_1 + \lambda_2 + 4\lambda_3 = 0 \\ 4\lambda_1 + 2\lambda_2 + 16\lambda_3 = 0 \end{cases}$$

$$\begin{cases} \lambda_1 = -\lambda_3 \\ \lambda_2 = -2\lambda_3 \\ 4\lambda_3 = 0 \end{cases}$$

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{cases}$$

→ linear onafhankelijk.

extra af 4

1.  $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$

2. \*  $\lambda_1 \begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix} + \lambda_2 \begin{pmatrix} a_2 & b_2 \\ b_2 & c_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 a_1 + \lambda_2 a_2 & \lambda_1 b_1 + \lambda_2 b_2 \\ \lambda_1 b_1 + \lambda_2 b_2 & \lambda_1 c_1 + \lambda_2 c_2 \end{pmatrix} \in S$

\*  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in S$

3.  $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

4.  $\left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R} \right\}$

### 4.3.13 Opdrachten

$$1. a) \begin{cases} \lambda + 2\mu + \gamma = 0 \\ 2\lambda - \mu + 7\gamma = 0 \\ \mu - \gamma = 0 \end{cases} \quad \begin{cases} -3\gamma + 2\gamma + \gamma = 0 \\ \lambda = -3\gamma \\ \mu = \gamma \end{cases}$$

$$\begin{cases} 0 = 0 \\ \lambda = -3\gamma \\ \mu = \gamma \end{cases} \rightarrow \text{lineair afhankelijk}$$

$$b) \begin{cases} \lambda + 3\mu + \gamma + 3\delta + 2\alpha = 0 \\ \lambda + \mu - \gamma - \delta + \alpha = 0 \\ -\lambda - \mu + \gamma + \delta - \alpha = 0 \\ 2\lambda + \mu - 3\gamma - 4\delta = 0 \end{cases}$$

$$\begin{cases} -5\lambda + 10\gamma + 15\delta + 2\alpha = 0 \\ -\lambda + 2\gamma + 3\delta + \alpha = 0 \\ \lambda - 2\gamma - 3\delta - \alpha = 0 \\ \mu = 3\gamma + 4\delta - 2\lambda \end{cases}$$

$$\begin{cases} -3\alpha = 0 \\ 0 = 0 \\ \lambda = 2\gamma + 3\delta + \alpha \\ \mu = 3\gamma + 4\delta - 2\lambda \end{cases} \quad \begin{cases} \alpha = 0 \\ 0 = 0 \\ \lambda = 2\gamma + 3\delta \\ \mu = -\gamma - 2\delta \end{cases}$$

$\rightarrow$  lineair afhankelijk

$$c) \begin{array}{ccc} \text{voor } x=0 & \text{voor } x=1 & \text{voor } x=2 \\ (1, 1, 0) & (2, 2, 2) & (3, 5, 6) \\ \begin{cases} \lambda + 2\mu + 3\gamma = 0 \\ \lambda + 2\mu + 5\gamma = 0 \\ 2\mu + 6\gamma = 0 \end{cases} & & \begin{cases} \gamma = 0 \\ \mu = 0 \\ \lambda = 0 \end{cases} \end{array}$$

$$d) \begin{cases} \lambda - \mu + 5\gamma = 0 \\ 2\lambda + 2\gamma = 0 \\ 3\lambda + 3\mu + 8\gamma = 0 \\ 4\lambda + \mu = 0 \end{cases} \quad \dots \quad \begin{cases} \lambda = 0 \\ \gamma = \frac{0}{2} \lambda \\ \mu = -4\lambda \end{cases}$$

$$\begin{cases} \lambda = 0 \\ \gamma = 0 \\ \mu = 0 \end{cases}$$

2. a) stel  $x = \frac{\pi}{2}$

$$\rightarrow \lambda \sin\left(\frac{\pi}{2}\right) + \mu \sin \pi + \gamma \sin \frac{3\pi}{2} = 0$$

afleiden en  $x=0$  invullen

$$\rightarrow \lambda \cos(0) + \mu 2 \cos(0) + \gamma 3 \cos(0) = 0$$

2x afleiden en  $x = \frac{\pi}{2}$  invullen

$$\rightarrow \lambda - \sin\left(\frac{\pi}{2}\right) + \mu - 4 \sin \pi + \gamma - 9 \sin\left(\frac{3\pi}{2}\right) = 0$$

$$\begin{cases} \lambda - \gamma = 0 \\ \lambda + 2\mu + 3\gamma = 0 \\ -\lambda + 9\gamma = 0 \end{cases} \quad \begin{cases} \lambda = 0 \\ \mu = 0 \\ \gamma = 0 \end{cases}$$

$\rightarrow$  lineair onafhankelijk

b) stel  $x=0$

$$\rightarrow \lambda e^0 \cos(0) + \mu e^0 \sin(0) = 0$$

stel  $x = \frac{\pi}{2}$

$$\rightarrow \lambda e^{\frac{\pi}{2}} \cos \frac{\pi}{2} + \mu e^{\frac{\pi}{2}} \sin\left(\frac{\pi}{2}\right) = 0$$

$$\begin{cases} \lambda = 0 \\ \mu e^{\frac{\pi}{2}} = 0 \end{cases} \quad \begin{cases} \lambda = 0 \\ \mu = 0 \end{cases}$$

$\rightarrow$  lineair onafhankelijk

c) lineair afhankelijk want  $\cos 2x = \cos^2 x - \sin^2 x$

3. a) Stel  $n = 0$

$$\rightarrow \lambda + \frac{\mu}{2} + \frac{\gamma}{3} = 0$$

Stel  $n = 1$

$$\frac{\lambda}{2} + \frac{\mu}{3} + \frac{\gamma}{4} = 0$$

Stel  $n = 2$

$$\frac{\lambda}{3} + \frac{\mu}{4} + \frac{\gamma}{5} = 0$$

$$\det \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} = \left(\frac{1}{15} - \frac{1}{16}\right) - \frac{1}{2} \left(\frac{1}{10} - \frac{1}{12}\right) + \frac{1}{3} \left(\frac{1}{8} - \frac{1}{9}\right)$$
$$= 0,000462963$$

$\rightarrow$  de enige opi van stelsel =  $(0, 0, 0)$

$\rightarrow$  lineair onafhankelijk.

b) lineair afhankelijk want

$$\frac{\lambda}{n+1} + \frac{\mu}{n+2} = \frac{\gamma}{(n+1)(n+2)}$$

$$\lambda(n+2) + \mu(n+1) = \gamma$$

voor  $\lambda = 1, \mu = -1, \gamma = 1$  klopt dit.

c) Stel  $n = 0$

$$\rightarrow \lambda + \mu + \gamma = 0$$

Stel  $n = 1$

$$\rightarrow \frac{\lambda}{2} + \frac{\mu}{4} + \frac{\gamma}{8} = 0$$

Stel  $n = 2$

$$\rightarrow \frac{\lambda}{3} + \frac{\mu}{9} + \frac{\gamma}{27} = 0$$

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 1/2 & 1/4 & 1/8 \\ 1/3 & 1/9 & 1/27 \end{bmatrix} = \left(\frac{1}{108} - \frac{1}{72}\right) + \left(\frac{1}{18} - \frac{1}{12}\right) - \left(\frac{1}{54} - \frac{1}{24}\right)$$
$$= \frac{-1}{108} \neq 0 \rightarrow \text{lineair onafhankelijk.}$$

4. a) de dimensie is  $m \times n$

Bewijs:

We tonen aan dat

$$\left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 \\ & \ddots & \\ 0 & \dots & 1 \end{pmatrix} \right\} \text{ een basis is van } (\mathbb{R}, \mathbb{R}^{m \times n}, +)$$

• voortbrengend

neem een willekeurige  $m \times n$  matrix  $A$

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix}. \text{ deze valt te schrijven als}$$

lineaire combinatie van onze potentiële basis

op de volgende manier:

$$a_{11} \begin{pmatrix} 1 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \end{pmatrix} + \dots + a_{mn} \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 1 \end{pmatrix}$$

• vrij

veronderstel dat

$$\lambda_{11} \begin{pmatrix} 1 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \end{pmatrix} + \dots + \lambda_{mn} \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 1 \end{pmatrix} = 0$$

Daaruit volgt het volgende stelsel

$$\begin{cases} \lambda_{11} + 0\lambda_{12} + \dots + 0\lambda_{mn} \\ \vdots \\ 0\lambda_{11} + 0\lambda_{12} + \dots + \lambda_{mn} \end{cases} \Leftrightarrow \begin{cases} \lambda_{11} = 0 \\ \lambda_{12} = 0 \\ \vdots \\ \lambda_{mn} = 0 \end{cases}$$

→ vrij deel.

→ basis

b) De dimensie is oneindig. we tonen dit aan met een bewijs uit het ongerijmde

Veronderstel dat de dimensie =  $n$

Dan kan de continue functie

$$f: \mathbb{R} \rightarrow \mathbb{R}: x \rightarrow 1 + x + x^2 + x^3 + \dots + x^n + x^{n+1}$$

nooit gemaakt worden.

5. Omdat  $\dim(\mathbb{R}, \mathbb{R}^4, +) = 4$  moeten we aantonen dat  $\{(0, 2, 4, 1), (1, -1, 3, 1), (1, 5, 5, 1), (0, 8, -4, 1)\}$  vrij is.

$$\begin{cases} \mu + \gamma = 0 \\ 2\lambda - \mu + 5\gamma + 8\alpha = 0 \\ 4\lambda + 3\mu + 5\gamma - 4\alpha = 0 \\ \lambda + \mu + \gamma + \alpha = 0 \end{cases} \Leftrightarrow \begin{cases} \mu = -\gamma \\ 2\lambda + 6\gamma + 8\alpha = 0 \\ 4\lambda + 2\gamma - 4\alpha = 0 \\ \lambda + \alpha = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \mu = -\gamma \\ 6\alpha + 6\gamma = 0 \\ -8\alpha + 2\gamma = 0 \\ \lambda = -\alpha \end{cases} \Leftrightarrow \begin{cases} \mu = -\gamma \\ \alpha = -\gamma \\ \gamma = 0 \\ \lambda = -\alpha \end{cases} \Leftrightarrow \begin{cases} \mu = 0 \\ \alpha = 0 \\ \gamma = 0 \\ \lambda = 0 \end{cases}$$

We lossen volgende stelsel op:

$$\begin{cases} \mu + \gamma = 1 \\ 2\lambda - \mu + 5\gamma + 8\alpha = 0 \\ 4\lambda + 3\mu + 5\gamma - 4\alpha = 0 \\ \lambda + \mu + \gamma + \alpha = 0 \end{cases} \quad \begin{cases} \mu = 1 - \gamma \\ 2\lambda + 6\gamma + 8\alpha - 1 = 0 \\ 4\lambda + 2\gamma - 4\alpha + 3 = 0 \\ \lambda + \alpha + 1 = 0 \end{cases}$$

$$\begin{cases} \mu = 1 - \gamma \\ 6\alpha + 6\gamma - 3 = 0 \\ -8\alpha + 2\gamma - 1 = 0 \\ \lambda = -\alpha - 1 \end{cases} \quad \begin{cases} \mu = 1 - \gamma \\ \alpha = -\gamma + 1/2 \\ +8\gamma - 4 + 2\gamma - 1 = 0 \\ \lambda = -\alpha - 1 \end{cases}$$

$$\begin{cases} \mu = 1/2 \\ \alpha = 0 \\ \gamma = 1/2 \\ \lambda = -1 \end{cases}$$

De coördinaten t.o.v. de basis zijn dus  $(-1, \frac{1}{2}, \frac{1}{2}, 0)$

6. Veranderstel

$$\begin{cases} 4\lambda + \mu + 3\gamma = 0 \\ 4\lambda - \mu + \gamma = 0 \\ -4\lambda + 2\mu = 0 \end{cases}$$

$$\begin{cases} 6\lambda + 3\gamma = 0 \\ 2\lambda + \gamma = 0 \\ \mu = 2\lambda \end{cases}$$

→ lineair afhankelijk

bv.  $\lambda = 1, \mu = 2, \gamma = -2$

dus  $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

dus  $\text{vect} \{ (4, 4, -4), (1, -1, 2) \} = \text{vect} \{ (4, 4, -4), (1, -1, 2), (3, 1, 0) \}$

Veranderstel

$$\begin{cases} 4\lambda + \mu = 0 \\ 4\lambda - \mu = 0 \\ -4\lambda + 2\mu = 0 \end{cases} \quad \begin{cases} \mu = 0 \\ \lambda = 0 \end{cases}$$

→ lineair onafhankelijk

→ dimensie = 2 met basis

$$\{ (1, 1, -1), (1, -1, 2) \}$$

$$\begin{cases} \lambda + \mu = 1 \\ \lambda - \mu = 0 \\ -\lambda + 2\mu = 0 \end{cases} \Leftrightarrow \begin{cases} \mu = 1/2 \\ \lambda = \mu \\ \mu = 0 \end{cases} \rightarrow \text{Strijdig}$$

$$\begin{cases} \lambda + \mu = -4 \\ \lambda - \mu = -10 \\ -\lambda + 2\mu = 13 \end{cases} \Leftrightarrow \begin{cases} \lambda = -4 - \mu \\ \lambda = -10 + \mu \\ 4 + \mu + 2\mu = 13 \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda = -7 \\ \mu = 3 \end{cases} \rightarrow \text{coördinaten t.o.v. de basis zijn } (-7, 3)$$

7. veronderstel

$$\begin{cases} \lambda + 3\mu + 3\gamma = 0 \\ 3\lambda - \mu + 4\gamma = 0 \\ -\lambda + \mu - \gamma = 0 \end{cases} \quad \begin{cases} 4\lambda + 6\gamma = 0 \\ 2\lambda + 3\gamma = 0 \\ \mu = \lambda + \gamma \end{cases}$$

$$\begin{cases} \lambda = -\frac{3}{2}\gamma \\ \mu = \lambda + \gamma \end{cases}$$

bv.  $\lambda = 3, \gamma = -2, \mu = 1$

Dus  $\frac{3}{2} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

Dus  $\text{vect}\{(1, 3, -1), (3, -1, 1)\} = \text{vect}\{(1, 3, -1), (3, -1, 1), (3, 4, -1)\}$

→ basis 1 =  $\{(1, 3, -1), (3, -1, 1)\}$   
 basis 2 =  $\{(1, 3, -1), (3, 4, -1)\}$

8.  $\begin{cases} x + 2y - z = 0 \\ 2x + y + z = 0 \end{cases} \Leftrightarrow \begin{cases} z = x + 2y \\ 3x + 3y = 0 \end{cases} \Leftrightarrow \begin{cases} z = y \\ x = -y \end{cases}$

De punten zijn dus van de verzameling:

$V = \{(-\lambda, \lambda, \lambda)\}$

Nu,  $a(-\lambda_1, \lambda_1, \lambda_1) + b(-\lambda_2, \lambda_2, \lambda_2)$

$= (-a\lambda_1 - b\lambda_2, a\lambda_1 + b\lambda_2, a\lambda_1 + b\lambda_2) \in V$

dimensie = 1 en een basis is  $(-1, 1, 1)$

9. geg:  $\lambda x + \mu y + \gamma z = 0 \Leftrightarrow \lambda = 0 \wedge \mu = 0 \wedge \gamma = 0$

TB:  $\lambda(x+y) + \mu(y+z) + \gamma(z+x) = 0$

$\Leftrightarrow \lambda = 0 \wedge \mu = 0 \wedge \gamma = 0$

Bewijs:  $(\lambda + \gamma)x + (\lambda + \mu)y + (\mu + \gamma)z = 0$

Dit kan volgens het gegeven alleen als

$$\begin{cases} \lambda + \gamma = 0 \\ \lambda + \mu = 0 \\ \mu + \gamma = 0 \end{cases}$$

$$\begin{cases} \lambda = -\gamma \\ -2\gamma = 0 \\ \mu = -\gamma \end{cases}$$

$$\begin{cases} \lambda = 0 \\ \gamma = 0 \\ \mu = 0 \end{cases}$$

$$10. a) \{ (1, 0, 0), (0, 1, 0) \}$$

$$e) \{ (1, 2, 0), (0, 0, 1) \}$$

$$f) \{ (2, 1, 0), (3, 0, 1) \}$$

$$g) \{ (-1, 1, 0), (-1, 0, 1) \}$$

## Extra oefeningen opdracht 3

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extra oefening 1

$$1. f \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = f(M_2) = 0 \cdot (1+x) + 2(1-x) + 3x^2 \\ = 2 - 2x + 3x^2$$

$$2. \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = M_1 + 0 M_2 + 0 M_3 + M_4$$

$$f \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = f \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + f \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \\ = 1 + x - 2x^2 + 1 + x + 1 - x + x^2 \\ = 3 + x - x^2$$

$$3. \text{NEEN } a) f(3M_1 - M_3) = 0 \text{ maar } 3M_1 - M_3 \neq 0$$

$$b) f(3M_1) = f(M_3) \text{ maar } 3M_1 \neq M_3$$

extra oef 2.

$$1. T \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (-2n^2 - n)_{n \in \mathbb{N}} \\ \lim_{n \rightarrow \infty} (-2n^2 - n) = -\infty$$

$$2. \left\{ \begin{pmatrix} 0 & b & c \\ 0 & -b & c \end{pmatrix} \text{ met } b, c \in \mathbb{R} \right\}$$

$$\text{basis} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$$3. \text{basis} = \left\{ (2)_{n \in \mathbb{N}}, (n)_{n \in \mathbb{N}}, (n^2)_{n \in \mathbb{N}}, (n^3)_{n \in \mathbb{N}} \right\}$$

## 4.4.9 opdrachten

1. a)  $f_1(\lambda_1 x_1 + \lambda_2 x_2) = 3\lambda_1 x_1 + 3\lambda_2 x_2 = \lambda_1 f_1(x_1) + \lambda_2 f_1(x_2)$   
 matrixvoorstelling  $\rightarrow (3)$

b)  $f_2(\lambda_1 x_1 + \lambda_2 x_2) = 2\lambda_1 x_1 + 2\lambda_2 x_2 + 1 \neq \lambda_1 f_2(x_1) + \lambda_2 f_2(x_2)$

c)  $f_3(\lambda_1(x_1, y_1) + \lambda_2(x_2, y_2)) = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_1 y_1 + \lambda_2 y_2 =$   
 $= \lambda_1 f_3(x_1, y_1) + \lambda_2 f_3(x_2, y_2)$   
 matrixvoorstelling  $\rightarrow (1, 1)$

d)  $f_4(\lambda_1(x_1, y_1, z_1) + \lambda_2(x_2, y_2, z_2))$   
 $= 5(\lambda_1 x_1 + \lambda_2 x_2) - 2(\lambda_1 y_1 + \lambda_2 y_2) + 3(\lambda_1 z_1 + \lambda_2 z_2)$   
 $= \lambda_1 f_4(x_1, y_1, z_1) + \lambda_2 f_4(x_2, y_2, z_2)$   
 matrixvoorstelling  $\rightarrow (5, -2, 3)$

e)  $f_5(\lambda_1(x_1, y_1) + \lambda_2(x_2, y_2))$   
 $= |(\lambda_1 x_1 + \lambda_2 x_2) - (\lambda_1 y_1 + \lambda_2 y_2)|$   
 $\neq \lambda_1 f_5(x_1, y_1) + \lambda_2 f_5(x_2, y_2)$

f)  $f_6(\lambda_1(x_1, y_1) + \lambda_2(x_2, y_2))$   
 $= (\lambda_1 x_1 + \lambda_2 x_2 - \lambda_1 y_1 - \lambda_2 y_2, \lambda_1 x_1 + \lambda_2 x_2 + \lambda_1 y_1 + \lambda_2 y_2)$   
 $= \lambda_1 f_6(x_1, y_1) + \lambda_2 f_6(x_2, y_2)$   
 matrixvoorstelling  $\rightarrow \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

h)  $f_8(\lambda_1(x_1, y_1, z_1) + \lambda_2(x_2, y_2, z_2))$   
 $= (\lambda_1 x_1 + \lambda_2 x_2 + \lambda_1 y_1 + \lambda_2 y_2 + \lambda_1 z_1 + \lambda_2 z_2, \lambda_1 x_1 + \lambda_2 x_2 + \lambda_1 y_1 + \lambda_2 y_2$   
 $- \lambda_1 z_1 - \lambda_2 z_2, \lambda_1 x_1 + \lambda_2 x_2 - \lambda_1 y_1 - \lambda_2 y_2 + \lambda_1 z_1 + \lambda_2 z_2, \lambda_1 x_1 + \lambda_2 x_2 -$   
 $\lambda_1 y_1 - \lambda_2 y_2 - \lambda_1 z_1 - \lambda_2 z_2)$

$= \lambda_1 f_8(x_1, y_1, z_1) + \lambda_2 f_8(x_2, y_2, z_2)$

matrixvoorstelling  $\rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$

$$\begin{aligned}
 \text{i)} \quad & f_g(\lambda_1(x_1, y_1, z_1) + \lambda_2(x_2, y_2, z_2)) \\
 &= (\lambda_1 z_1 + \lambda_2 z_2 + 1, (\lambda_1 x_1 + \lambda_2 x_2)^2) \\
 &\neq \lambda_1 f_g(x_1, y_1, z_1) + \lambda_2 f_g(x_2, y_2, z_2)
 \end{aligned}$$

$$\begin{aligned}
 \text{g)} \quad & f_7(\lambda_1(x_1, y_1) + \lambda_2(x_2, y_2)) \\
 &= (\lambda_1 x_1 + \lambda_2 x_2 + 2(\lambda_1 y_1 + \lambda_2 y_2), 1) \\
 &\neq \lambda_1 f_7(x_1, y_1) + \lambda_2 f_7(x_2, y_2)
 \end{aligned}$$

$$\begin{aligned}
 \text{j)} \quad & f_{10}(\lambda_1(x_1, y_1, z_1) + \lambda_2(x_2, y_2, z_2)) \\
 &= (\lambda_1 z_1 + \lambda_2 z_2 + 3(\lambda_1 y_1 + \lambda_2 y_2), \lambda_1 x_1 + \lambda_2 x_2 - \lambda_1 y_1 - \lambda_2 y_2 \\
 &\quad - \lambda_1 z_1 - \lambda_2 z_2) \\
 &= \lambda_1 f_{10}(x_1, y_1, z_1) + \lambda_2 f_{10}(x_2, y_2, z_2)
 \end{aligned}$$

$$\text{matrix voorstelling} \rightarrow \begin{pmatrix} 0 & 3 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\begin{aligned}
 \text{k)} \quad & f_{11}(\lambda_1(x_1, y_1) + \lambda_2(x_2, y_2)) \\
 &= (\sin(\lambda_1 x_1 + \lambda_2 x_2), 7(\lambda_1 y_1 + \lambda_2 y_2), (\lambda_1 x_1 + \lambda_2 x_2)(\lambda_1 y_1 + \lambda_2 y_2)) \\
 &\neq \lambda_1 f_{11}(x_1, y_1) + \lambda_2 f_{11}(x_2, y_2)
 \end{aligned}$$

$$\begin{aligned}
 \text{l)} \quad & f_{12}(\lambda_1(a_1 + b_1 x + c_1 x^2) + \lambda_2(a_2 + b_2 x + c_2 x^2)) \\
 &= \lambda_1 c_1 + \lambda_2 c_2 + 2(\lambda_1 a_1 + \lambda_2 a_2)x + (\lambda_1 a_1 + \lambda_2 a_2 + \lambda_1 b_1 + \\
 &\quad \lambda_2 b_2 + \lambda_1 c_1 + \lambda_2 c_2)x^3
 \end{aligned}$$

$$= \lambda_1 f_{12}(a_1 + b_1 x + c_1 x^2) + \lambda_2 f_{12}(a_2 + b_2 x + c_2 x^2)$$

$$\text{matrix voorstelling} \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\text{m)} \quad f_{13}\left(\lambda_1 \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \lambda_2 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right)$$

$$= (\lambda_1 a_1 + \lambda_2 a_2 + \lambda_1 c_1 + \lambda_2 c_2, \lambda_1 b_1 + \lambda_2 b_2 - \lambda_1 d_1 - \lambda_2 d_2)$$

$$= \lambda_1 f_{13}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \lambda_2 f_{13}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

$$\text{matrix voorstelling} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

2. t.o.v. standaardbasis:

$$\begin{pmatrix} -3 & -2 & 2 \\ -13 & -5 & 7 \\ -17 & -8 & 10 \end{pmatrix}$$

t.o.v. gegeven basis:

① basis invullen in de functie

$$\rightarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & -2 & 2 \\ 1 & -3 & 2 \end{pmatrix}$$

② als lineaire combinatie schrijven t.o.v. gegeven basis

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

3.  $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

\*  $f(1, 2) = (0, -1)$

$$\rightarrow a_{11} + 2a_{12} = 0$$

$$a_{21} + 2a_{22} = -1$$

\*  $f(-1, 1) = (2, 1)$

$$\rightarrow -a_{11} + a_{12} = 2$$

$$-a_{21} + a_{22} = 1$$

$$\begin{cases} a_{11} + 2a_{12} = 0 \\ a_{21} + 2a_{22} = -1 \\ -a_{11} + a_{12} = 2 \\ -a_{21} + a_{22} = 1 \end{cases} \Leftrightarrow \begin{cases} a_{11} = -2a_{12} \\ a_{21} = -1 - 2a_{22} \\ 3a_{12} = 2 \\ 3a_{22} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} a_{11} = -4/3 \\ a_{21} = -1 \\ a_{12} = 2/3 \\ a_{22} = 0 \end{cases} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -4/3 & 2/3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\rightarrow f(x,y) = \left( \frac{-4x+2y}{3}, -x \right)$$

4.  $f(\lambda_1(a_1, b_1, c_1) + \lambda_2(a_2, b_2, c_2))$

$$= \lambda_1 c_1 + \lambda_2 c_2 + (\lambda_1 a_1 + \lambda_2 a_2 + \lambda_1 b_1 + \lambda_2 a_2) x$$

$$+ (\lambda_1 a_1 + \lambda_2 a_2 + \lambda_1 c_1 + \lambda_2 c_2) x^2 + (\lambda_1 b_1 + \lambda_2 b_2 + \lambda_1 c_1 + \lambda_2 c_2) x^3$$

$$= \lambda_1 (c_1 + (a_1 + b_1)x + (a_1 + c_1)x^2 + (b_1 + c_1)x^3)$$

$$+ \lambda_2 (c_2 + (a_2 + b_2)x + (a_2 + c_2)x^2 + (b_2 + c_2)x^3)$$

$$= \lambda_1 f(a_1, b_1, c_1) + \lambda_2 f(a_2, b_2, c_2)$$

b1)

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

b2) ① tov.  $\{1+x, 1+x^2, x+x^2\}$  en standaardbasis

$$\begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

② tov.  $\{1+x, 1+x^2, x+x^2\}$  en  $\{1, 1+x, 1+x^2, 1+x^3\}$

$$\begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix} = -4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

→ matrixvoorstelling wordt dus

$$\begin{pmatrix} -4 & -3 & -3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

5. a) Omdat  $\mathbb{R}^{2 \times 2}$  als dimensie 4 heeft moeten we enkel aantonen dat B vrij is.

veronderstel dat

$$\lambda_1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} + \lambda_4 \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

we tonen aan dat  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$

$$\begin{cases} \lambda_4 = 0 \\ \lambda_2 + 2\lambda_3 + \lambda_4 = 0 \\ \lambda_1 + 2\lambda_2 + 3\lambda_3 - \lambda_4 = 0 \\ \lambda_3 + \lambda_4 = 0 \end{cases} \iff \begin{cases} \lambda_4 = 0 \\ \lambda_2 = 0 \\ \lambda_1 = 0 \\ \lambda_3 = 0 \end{cases}$$

$$b) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\text{dus } f \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 1 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 2 \\ 5 \end{bmatrix} \text{ volgens B}$$

$$f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 5 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} + 5 \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 13 \\ 14 & 7 \end{pmatrix}$$

6. a) + Lemma:

veronderstel dat  $L$  een lineaire afbeelding is, dan is  $L(0) = 0$ .

Bewijs:

Omdat  $L$  lineair is geldt:

$$\lambda L(a) = L(\lambda a) \text{ voor alle } \lambda \in \mathbb{R}$$

Stel nu  $\lambda = 0$ . Dan geldt:

$$0 \cdot L(a) = L(0)$$

$$0 = L(0)$$

\* Het echte bewijs:

$$\text{geg: als } \lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_n e_n = 0$$

$$\text{dan is } \lambda_1 = \lambda_2 = \lambda_3 = \dots = \lambda_n = 0$$

$$\text{TB: als } \lambda_1 L(e_1) + \lambda_2 L(e_2) + \dots + \lambda_n L(e_n) = 0$$

$$\text{dan is } \lambda_1 = \lambda_2 = \lambda_3 = \dots = \lambda_n = 0$$

$$\text{Bewijs: } \lambda_1 L(e_1) + \lambda_2 L(e_2) + \dots + \lambda_n L(e_n) = 0$$

$$\Leftrightarrow L(\lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_n e_n) = 0$$

Nu weten we ook dat  $L(0) = 0$ .

Nu is een afbeelding injectief als

$$L(x_1) = L(x_2) \Leftrightarrow x_1 = x_2$$

$$\text{Of nog } \lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_n e_n = 0$$

en dus volgens het gegeven geldt dan

$$\lambda_1 = \lambda_2 = \lambda_3 = \dots = \lambda_n = 0$$

b) Beschouw de afbeelding  $L: \mathbb{R}^2 \rightarrow \mathbb{R}; (x, y) \rightarrow x$

$\{(1, 0), (0, 1)\}$  is een vrij deel van  $\mathbb{R}^2$

maar  $\{L(1, 0) = 1, L(0, 1) = 0\}$  is geen vrij deel van  $\mathbb{R}$

## 4.5.5 opdrachten

1. a) • homogene vgl:

$$y_{n+1} - y_n = 0$$

$$y_{n+1} = y_n$$

$$\Rightarrow y_n = 1^n \cdot C = C \quad \text{met } C \in \mathbb{R}$$

• 1 particuliere oplossing

we stellen een opl. v.d. vorm  $\alpha e^n$  voor met  $\alpha \in \mathbb{R}$   
invullen geeft

$$\alpha e^{n+1} - \alpha e^n = e^n$$

$$\Leftrightarrow \alpha e - \alpha = 1$$

$$\Leftrightarrow \alpha = \frac{1}{e-1}$$

• alle oplossingen zijn v.d. vorm

$$y_n = \frac{1}{e-1} e^n + C$$

b) • homogene vgl:

$$y_{n+1} + 3y_n = 0$$

$$\Rightarrow y_n = C \cdot (-3)^n \quad \text{met } C \in \mathbb{R}.$$

• 1 particuliere opl:

→ constante rij 1

• alle oplossingen zijn v.d. vorm

$$C \cdot (-3)^n + 1$$

c) • homogene vgl:

$$2y_{n+1} - y_n = 0$$

$$y_{n+1} - \frac{1}{2}y_n = 0$$

$$\rightarrow y_n = \left(\frac{1}{2}\right)^n \cdot C \quad \text{met } C \in \mathbb{R}$$

• particuliere opl:

$$y_{n+1} - \frac{1}{2}y_n = 3$$

→ constante rij 6

• alle oplossingen:  $C \cdot \left(\frac{1}{2}\right)^n + 6$  met  $C \in \mathbb{R}$ .

d) • homogene vgl

$$y_{n+1} - 0,2y_n = 0$$

$$\rightarrow y_n = \left(\frac{1}{5}\right)^n \cdot C \quad \text{met } C \in \mathbb{R}$$

• particuliere oplossing

$$\rightarrow \text{constante rij } 5$$

• alle oplossingen zijn vid vorm

$$C \cdot \left(\frac{1}{5}\right)^n + 5 \quad \text{met } C \in \mathbb{R}$$

e) • homogene vgl

$$y_{n+1} - 2y_n = 0$$

$$\rightarrow y_n = 2^n \cdot C$$

• particuliere oplossing

voorstel vid vorm  $\alpha n + \beta$  met  $\alpha, \beta \in \mathbb{R}$

$$\text{invullen: } \alpha(n+1) + \beta - 2\alpha n - 2\beta = n$$

$$-\alpha n + \alpha - \beta = n$$

$$\begin{cases} -\alpha = 1 \\ \alpha - \beta = 0 \end{cases} \quad \begin{cases} \alpha = -1 \\ \beta = -1 \end{cases}$$

• algemere oplossing:

$$C \cdot 2^n - n - 1$$

f) • homogene vgl

$$y_{n+1} = (n+1)y_n = (n+1)n y_{n-1} = \dots = (n+1)! y_0$$

$$y_n = C \cdot n!$$

• particuliere oplossing

$$y_{n+1} = 1 + (n+1)y_n$$

$$\begin{aligned} y_n &= 1 + n y_{n-1} = 1 + n + n(n-1)y_{n-2} \\ &= 1 + n + n(n-1) + n(n-1)(n-2)y_{n-3} \\ &= 1 + n + n(n-1) + \dots + n! \cdot y_0 \\ &= \frac{n!}{n!} + \frac{n!}{(n-1)!} + \dots + \frac{n!}{1!} \\ &= n! \sum_{k=0}^n \frac{1}{k!} \end{aligned}$$

• algemere oplossing:  $y_n = C \cdot n! + n! \sum_{k=0}^n \frac{1}{k!}$

g) • homogene vgl = algemere vgl

$$y_{n+1} = e^{2n} y_n$$

$$y_n = e^{2(n-1)} y_{n-1} = e^{2(n-1)+2(n-2)} y_{n-2}$$

$$= e^{2(n^2 - \sum_{k=1}^n k)} \cdot C \quad \text{met } C \in \mathbb{R}$$

h) • homogene vgl

$$(n+2) y_{n+1} - (n+1) y_n = 0$$

$$y_{n+1} = \frac{n+1}{n+2} y_n$$

$$y_n = \frac{n}{n+1} y_{n-1} = \frac{n}{n+1} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdots \frac{1}{2} \cdot C$$
$$= \frac{C}{n+1}$$

• 1 particuliere ope

$$(n+2) y_{n+1} - (n+1) y_n = n+1$$

$$y_{n+1} = \frac{n+1}{n+2} + \frac{n+1}{n+2} y_n$$

$$y_n = \frac{n}{n+1} + \frac{n}{n+1} \left( \frac{n-1}{n} + \frac{n-1}{n} y_{n-2} \right)$$

$$= \frac{n}{n+1} + \frac{n}{n+1} \left( \frac{n-1}{n} + \frac{n-1}{n} \left( \frac{n-2}{n-1} + \frac{n-2}{n-1} y_{n-3} \right) \right)$$

$$= \frac{n}{n+1} + \frac{n^2 - \sum_{k=0}^n k}{n+1}$$

$$= n - \frac{\sum_{k=0}^n k}{n+1}$$

2. • homogene vgl

voorstel  $\lambda^n$

$$\lambda^{n+1} - (1-2p) \lambda^n = 0$$

$$\lambda - 1 + 2p = 0$$

$$\lambda = 1 - 2p$$

$$y_n = (1-2p)^n \cdot C$$

• 1 particuliere oplossing

$$y_{n+1} = p + (1-2p)y_n$$

$$y_n = p + (1-2p)y_{n-1}$$

$$= p + p(1-2p) + p(1-2p)^2 + p(1-2p)^3 + \dots + (1-2p)^{n-1} p$$
$$= (1-2p)^{n-1} + p$$

• algemere oplossing

$$y_n = (1-2p)^{n-1} + p \sum_{k=0}^{n-2} (1-2p)^k + (1-2p)^n \cdot C$$

• 1 particuliere oplossing

voorstel:  $\alpha$

$$\text{invullen: } \alpha - (1-2p)\alpha = p$$

$$\alpha(2p) = p$$

$$\alpha = 1/2$$

• algemere oplossing

$$y_n = (1-2p)^n \cdot C + \frac{1}{2}$$

• beginvoorwaarde

$$y_0 = 1-p = C + \frac{1}{2}$$

$$C = \frac{1}{2} - p$$

• oplossing wordt:

$$y_n = (1-2p)^n \left( \frac{1}{2} - p \right) + \frac{1}{2}$$

$$= \frac{1}{2} (1-2p)^{n+1} + \frac{1}{2}$$

$$= \frac{1}{2} \left( (1-2p)^n + 1 \right)$$

b) • homogene vge:

voorstel  $\lambda^n$

$$\lambda^{n+1} - (n+1)\lambda^n = 0$$

$$\lambda \cdot (n+1) = 0$$

$$\lambda = n+1$$

$$y_n = (n+1)^n \cdot C$$

• 1 particuliere oplossing

$$\begin{aligned}y_n &= n! + n y_{n-1} \\ &= n! + n((n-1)! + (n-1)y_{n-2}) \\ &= n! + n! + n(n-1)y_{n-2} \\ &= n! + n! + n! + \dots + n! \\ &= (n+1)n! = (n+1)!\end{aligned}$$

• algemene oplossing

$$y_n = (n+1)^n \cdot C + (n+1)!$$

• beginvoorwaarde:

$$y_0 = 1 = C + 1$$

$$\Leftrightarrow C = 0$$

$$y_n = (n+1)!$$

3. a)  $y_{n+1} = y_n \left(1 + \frac{0,08}{12}\right) - A$  met  $y_0 = 75000$

b) • homogene vgl

$$y_n = \left(1 + \frac{0,08}{12}\right)^n \cdot C \quad \text{met } C \in \mathbb{R}$$

• 1 particuliere oplossing

voorstel:  $\alpha$

$$\begin{aligned}\text{invullen: } \alpha &= \left(1 + \frac{0,08}{12}\right) \alpha - A \\ \alpha &= 150A\end{aligned}$$

• algemene oplossing:

$$y_n = 150A + \left(1,006666\right)^n \cdot C \quad \text{met } C \in \mathbb{R}$$

• beginvoorwaarde

$$y_0 = 75000 = 150A + 1 \cdot C$$

$$C = 75000 - 150A$$

$$y_n = 150A + \left(1,006666\right)^n \cdot (75000 - 150A)$$

c)  $y_{240} = 0 = 150A + \left(1 + \frac{0,08}{12}\right)^{240} \cdot (75000 - 150A)$   
 $A = 627,33$

4.  $K_{n+1} = K_n \cdot 1,06 - 20000$  met  $K_0 = K$

b) - homogene vgl:

$$\# K_n = (1,06)^n \cdot C \quad \text{met } C \in \mathbb{R}$$

• 1 particuliere oplossing

voorstel:  $\alpha$

$$\text{invullen: } \alpha = 1,06\alpha - 20000$$

$$\alpha = 333.333,333$$

• algemene oplossing

$$K_n = 333.333,33 + (1,06)^n \cdot C \quad \text{met } C \in \mathbb{R}$$

• beginvoorwaarde

$$K_0 = K = 333.333,333 + C$$

$$C = K - 333.333,333$$

$$K_n = 333.333,333 + (1,06)^n \cdot (K - 333.333,333)$$

c)  $K \geq 333.333,333$

d) a)  $K_{n+1} = K_n \cdot 1,06 - 20000 \cdot (1,02)^n$

b) - homogene vgl:

$$\# K_n = (1,06)^n \cdot C \quad \text{met } C \in \mathbb{R}$$

• 1 particuliere ope:

$$\text{voorstel: } \alpha (1,02)^n$$

$$\text{invullen: } \alpha (1,02)^{n+1} - \alpha (1,02)^n \cdot (1,06) = -20000 \cdot (1,02)^n$$

$$\Leftrightarrow \alpha (1,02) - \alpha \cdot (1,06) = -20000$$

$$\alpha = 500000$$

• algemene oplossing:

$$K_n = (1,06)^n \cdot C + 500000 \cdot (1,02)^n \quad \text{met } C \in \mathbb{R}$$

• beginvoorwaarde

$$K_0 = K = C + 500000$$

$$C = K - 500000$$

$$K_n = (1,06)^n \cdot (K - 500000) + 500000 \cdot (1,02)^n$$

c)  $K > 500.000$

5. hoogte 1 = 1 nodig  
 hoogte 2 = 3 nodig  $\cdot 2 + 1$   
 hoogte 3 = 7 nodig  $\cdot 2 + 1$   
 hoogte 4 = 15 nodig  $\cdot 2 + 1$   
 $a_n = 2a_{n-1} + 1$  met  $a_1 = 1$

• homogene vgl:

$$a_n = (2^n) \cdot C \text{ met } C \in \mathbb{R}$$

• 1 particuliere oplossing

$$x - 2x = 1$$

$$x = -1$$

• algemene oplossing:

$$a_n = -1 + 2^n \cdot C \text{ met } C \in \mathbb{R}$$

• beginvoorwaarde

$$a_1 = 1 = -1 + 2C$$

$$C = 1$$

$$\rightarrow a_n = 2^n - 1$$

$A, B \in \mathbb{R}$  6. a) homogene vgl = algemene vgl

• voorstel =  $\lambda^n$

$$\lambda^{n+2} - (2 \cos \varphi) \lambda^{n+1} + \lambda^n = 0$$

$$\lambda^2 - 2 \cos \varphi \lambda + 1 = 0$$

$$D = 4 \cos^2 \varphi - 4 \leq 0$$

$$\lambda = \frac{2 \cos \varphi \pm \sqrt{4 \cos^2 \varphi - 4}}{2}$$

$$= \cos \varphi \pm \sqrt{\cos^2 \varphi - 1}$$

$$= \cos \varphi \pm \sqrt{-\sin^2 \varphi}$$

⊕ als  $\sin^2 \varphi = 0 \Leftrightarrow \sin \varphi = 0$

$$\text{dan: } \lambda^2 - 2 \cos \varphi \lambda + 1 = (\lambda - \cos \varphi)^2$$

$$y_n^{(1)} = (\cos \varphi)^n$$

$$y_n^{(2)} = n(\cos \varphi)^n$$

de algemene oplossing is dan  $(A + Bn)(\cos \varphi)^n$

met  $A, B \in \mathbb{R}$

$$\textcircled{2} \text{ als } \sin^2 \varphi \neq 0 \Leftrightarrow \sin \varphi \neq 0$$

$$\lambda_1 = \cos \varphi + i \sin \varphi$$

$$\lambda_2 = \cos \varphi - i \sin \varphi$$

$$\tilde{y}_n^{(1)} = (\cos \varphi + i \sin \varphi)^n$$

$$\tilde{y}_n^{(2)} = (\cos \varphi - i \sin \varphi)^n$$

→ we maken de oplossingen reëel door de

juiste lineaire combinaties te nemen

$$y_n^{(1)} = \frac{1}{2} (\tilde{y}_n^{(1)} + \tilde{y}_n^{(2)}) = \frac{1}{2} ((\cos \varphi + i \sin \varphi)^n + (\cos \varphi - i \sin \varphi)^n)$$

$$y_n^{(2)} = \frac{1}{2i} (\tilde{y}_n^{(1)} - \tilde{y}_n^{(2)}) = \frac{1}{2i} ((\cos \varphi + i \sin \varphi)^n - (\cos \varphi - i \sin \varphi)^n)$$

$$\kappa = \sqrt{\cos^2 \varphi + \sin^2 \varphi} = 1$$

$$\cos \theta = \cos \varphi \quad \sin \theta = \sin \varphi$$

$$\Leftrightarrow \theta = \varphi$$

$$y_n^{(1)} = \frac{1}{2} (\kappa^n (\cos n\varphi + i \sin n\varphi) + \kappa^n (\cos n\varphi - i \sin n\varphi)) \\ = r^n \cos n\varphi = \cos n\varphi$$

$$y_n^{(2)} = \frac{1}{2i} (\kappa^n (\cos n\varphi - i \sin n\varphi) - \kappa^n (\cos n\varphi + i \sin n\varphi)) \\ = r^n \cdot \sin n\varphi = \sin n\varphi$$

de algemene oplossing is dus

$$y_n = A \cos n\varphi + B \sin n\varphi \text{ met } A, B \in \mathbb{R}$$

b) • homogene vgl:

$$y_{n+2} - 2y_{n+1} + 3y_n = 0$$

Karakteristieke vgl:

$$\lambda^2 - 2\lambda + 3 = 0$$

$$D = -8$$

$$\lambda = \frac{2 \pm 2\sqrt{2}i}{2} = 1 + \sqrt{2}i \vee 1 - \sqrt{2}i$$

$$\tilde{y}_n^{(1)} = (1 + \sqrt{2}i)^n \quad \tilde{y}_n^{(2)} = (1 - \sqrt{2}i)^n$$

→ oplossingen reëel maken

$$y_n^{(1)} = \frac{1}{2} (\tilde{y}_n^{(1)} + \tilde{y}_n^{(2)})$$

$$y_n^{(2)} = \frac{1}{2i} (\tilde{y}_n^{(1)} - \tilde{y}_n^{(2)})$$

Nu  $1 + \sqrt{2}i$   $r = \sqrt{1+2} = \sqrt{3}$

$\cos \theta = \frac{1}{\sqrt{3}}$   $\sin \theta = \frac{\sqrt{2}}{\sqrt{3}} \Rightarrow \theta =$

$y_n^{(1)} = \frac{1}{2} (\sqrt{3}^n (\cos n\theta + i \sin n\theta) + \sqrt{3}^n (\cos n\theta - i \sin n\theta))$   
 $= (\sqrt{3})^n \cos n\theta$

$y_n^{(2)} = \frac{1}{2i} (\sqrt{3}^n (\cos n\theta + i \sin n\theta) - \sqrt{3}^n (\cos n\theta - i \sin n\theta))$   
 $= (\sqrt{3})^n \sin n\theta$

$y_{n,n} = (\sqrt{3})^n (A \cos n\theta + B \sin n\theta)$  met  $\cos \theta = \frac{1}{\sqrt{3}}$  en  $\sin \theta = \frac{\sqrt{2}}{3}$

• particuliere oplossing:

$a - 2a + 3a = 4$

$a = 2$

dus  $y_{p,n} = 2$

• algemere oplossing:

$y_n = (\sqrt{3})^n (A \cos n\theta + B \sin n\theta) + 2$  met  $\cos \theta = \frac{1}{\sqrt{3}}$  en  $\sin \theta = \frac{\sqrt{2}}{3}$

c) • homogeen vgl:

$y_{n+2} - 2y_{n+1} + 4y_n = 0$

$D = 4 - 16 = -12$

$\lambda = \frac{2 \pm 2\sqrt{3}i}{2} = 1 + \sqrt{3}i \vee 1 - \sqrt{3}i$

$\tilde{y}_n^{(1)} = (1 + \sqrt{3}i)^n$   $\tilde{y}_n^{(2)} = (1 - \sqrt{3}i)^n$

→ we gaan ze reëel maken

$y_n^{(1)} = \frac{1}{2} (\tilde{y}_n^{(1)} + \tilde{y}_n^{(2)})$

$y_n^{(2)} = \frac{1}{2i} (\tilde{y}_n^{(1)} - \tilde{y}_n^{(2)})$

Nu,  $(1 + \sqrt{3}i)$   $r = \sqrt{1+3} = 2$

$\cos \theta = \frac{1}{2}$   $\sin \theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 60^\circ = \frac{\pi}{3} \text{ rad}$

Dus  $y_n^{(1)} = \frac{1}{2} (2^n (\cos n\frac{\pi}{3} + i \sin n\frac{\pi}{3}) + 2^n (\cos n\frac{\pi}{3} - i \sin n\frac{\pi}{3}))$   
 $= 2^n \cos n\frac{\pi}{3}$

$y_n^{(2)} = \frac{1}{2i} (2^n (\cos n\frac{\pi}{3} + i \sin n\frac{\pi}{3}) - 2^n (\cos n\frac{\pi}{3} - i \sin n\frac{\pi}{3}))$   
 $= 2^n \sin n\frac{\pi}{3}$

Dus  $y_{h,n} = 2^n (A \cos n\frac{\pi}{3} + B \sin n\frac{\pi}{3})$

• particuliere oplossing:  $y_{p,n} = \frac{4}{3}$

• algemere oplossing:  $y_n = 2^n (A \cos n\frac{\pi}{3} + B \sin n\frac{\pi}{3}) + \frac{4}{3}$

d) • homogene vgl

$$y_{n+2} - 4y_{n+1} + 4y_n = 0$$

voorstel:  $\lambda^n$

$$\lambda^{n+2} - 4\lambda^{n+1} + 4\lambda^n = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$D = 16 - 16 = 0$$

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$$

$$\lambda_1 = 2^n$$

$$\lambda_2 = U_n 2^n$$

$$\text{invullen: } U_{n+2} 2^{n+2} - 4U_{n+1} 2^{n+1} + 4U_n 2^n = 0$$

$$4U_{n+2} - 8U_{n+1} + 4U_n = 0$$

$U_n = n$  voldoet hieraan want

$$4(n+2) - 8(n+1) + 4n = 4n + 8 - 8n - 8 + 4n = 0$$

$$y_{h,n} = A 2^n + B n 2^n$$

• particuliere oplossing

$$y_{p,n} = 7$$

• algemere oplossing:

$$y_n = (A + Bn) 2^n + 7$$

e) • homogene vgl:

$$\lambda^2 - 4\lambda + 4 = 0$$

(zie hierboven)

$$y_{h,n} = A 2^n + B n \cdot 2^n$$

• particuliere oplossing:

voorstel:  $\alpha 2^n$

$$\text{invullen } \alpha 2^{n+2} - 4\alpha 2^{n+1} + 4\alpha 2^n = 2^n$$

$$\alpha \cdot 4 - 8\alpha + 4\alpha = 1$$

→ 2 is een wortel (dubbel)

→ voorstel:  $\alpha n^2 2^n$

$$\text{invullen: } (n+2)^2 \cdot \alpha \cdot 4 - (n+1)^2 \cdot 4 \cdot 2 \cdot \alpha + (n)^2 \cdot 4 \cdot \alpha = 1$$

$$(n^2 + 4n + 4) 4\alpha - (n^2 + 2n + 1) \cdot 8\alpha + 4\alpha n^2 = 1$$

$$8\alpha = 1$$

$$\alpha = 1/8$$

$$y_n = (A + nB) 2^n + \frac{1}{8} n^2 \cdot 2^n = 2^n \cdot \left( A + Bn + \frac{n^2}{8} \right)$$

f) • homogene vgl

$$\lambda^2 - 5\lambda + 6 = 0$$

$$D = 25 - 24 = 1$$

$$\lambda = \frac{5 \pm 1}{2} = 3 \vee 2$$

$$y_{h,n} = \tilde{A} \cdot 3^n + B \cdot 2^n$$

• 1 particuliere oplossing

voorstel:  $\alpha + \beta n$

$$\text{invullen: } \alpha + \beta(n+2) - 5\alpha - 5\beta(n+1) + 6\alpha + 6\beta n = 2 + 4n$$

$$\beta n - 5\beta n + 6\beta n + \alpha + 2\beta - 5\alpha - 5\beta + 6\alpha = 2 + 4n$$

$$\begin{cases} 2\alpha - 3\beta = 2 \\ 2\beta = 4 \end{cases} \quad \begin{cases} \alpha = 4 \\ \beta = 2 \end{cases}$$

$$y_{p,n} = 4 + 2n$$

• algemene oplossing:

$$y_n = A \cdot 3^n + B \cdot 2^n + 4 + 2n$$

g) • homogene vgl:

$$\lambda^2 + 2\lambda - 3 = 0$$

$$D = 4 + 12 = 16$$

$$\lambda = \frac{-2 \pm 4}{2} = -3 \vee 1$$

$$y_{h,n} = A + (-3)^n \cdot B$$

• 1 particuliere oplossing:

voorstel:  $\alpha + \beta n + \gamma n^2 + \delta n^3$

$$\text{invullen: } \alpha + \beta(n+2) + \gamma(n+2)^2 + \delta(n+2)^3 + 2(\alpha + \beta(n+1) + \gamma(n+1)^2 + \delta(n+1)^3) - 3(\alpha + \beta n + \gamma n^2 + \delta n^3) = n^3 + 1$$

$$\Leftrightarrow \alpha + \beta n + 2\beta + \gamma n^2 + \gamma 4n + 4\gamma^2 + \delta n^3 + 6\delta n^2 + 12\delta n + 8\delta + 2\alpha + 2\beta n + 2\beta + 2\gamma n^2 + 4\gamma n + 2\gamma + 2\delta n^3 + 6\delta n^2 + 6\delta n + 2\delta - 3\alpha - 3\beta n - 3\gamma n^2 - 3\delta n^3 = n^3 + 1$$

$\Leftrightarrow 0\delta = 1 \rightarrow$  strijdig  $\rightarrow$  alles maar  $n$  dan

voorstel:  $\alpha n + \beta n^2 + \gamma n^3 + \delta n^4$

invullen:  $\alpha n + 2\alpha + \beta n^2 + 4\beta n + 4\beta + \gamma n^3 + 6\gamma n^2 + 12\gamma n + 8\gamma + \delta n^4 + 8\delta n^3 + 24\delta n^2 + 32\delta n + 16\delta + 2\alpha n + 2\alpha + 2\beta n^2 + 4\beta n + 2\beta + 2\gamma n^3 + 6\gamma n^2 + 6\gamma n + 2\gamma + 2\delta n^4 + 8\delta n^3 + 12\delta n^2 + 8\delta n + 2\delta - 3\alpha n - 3\beta n^2 - 3\gamma n^3 - 3\delta n^4 = n^3 + 1$

$$\begin{cases} \gamma + 8\delta + 2\gamma + 8\delta - 3\gamma = 1 \\ \beta + 6\gamma + 24\delta + 2\beta + 6\gamma + 12\delta - 3\beta = 0 \\ \alpha + 4\beta + 12\gamma + 32\delta + 2\alpha + 4\beta + 6\gamma + 8\delta - 3\alpha = 0 \\ 2\alpha + 4\beta + 8\gamma + 16\delta + 2\alpha + 2\beta + 2\gamma + 2\delta = 1 \end{cases}$$

$$\begin{cases} \delta = 1/16 \\ \gamma = -3/16 \\ \beta = 7/64 \\ \alpha = 35/128 \end{cases}$$

$$y_{p,n} = \frac{35n}{128} + \frac{7n^2}{64} - \frac{3n^3}{16} + \frac{n^4}{16}$$

• algemere oplossing:

$$y_n = A + (-3)^n \cdot B + \frac{35n}{128} + \frac{7n^2}{64} - \frac{3n^3}{16} + \frac{n^4}{16}$$

h) • homogene vgl:

$$\lambda^2 - 3\lambda + 2 = 0$$

$$D = 9 - 8$$

$$\lambda = \frac{3 \pm 1}{2} = 2 \vee 1$$

$$y_{h,n} = A \cdot 2^n + B$$

• particuliere oplossing

voorstel:  $\alpha 4^n + \beta n^3 + \gamma n^2 + \delta n$  want 1 is wortel

invullen:  $\alpha 4^{n+2} + \beta(n+2)^3 + \gamma(n+2)^2 + \delta(n+2) - 3\alpha 4^{n+1} - 3\beta(n+1)^3 - 3\gamma(n+1)^2 - 3\delta(n+1) + 2\alpha 4^n + 2\beta n^3 + 2\gamma n^2 + 2\delta n = 4^n + 3n^2$

$$\begin{aligned}
 & 16\alpha \cdot 4^n + \beta n^3 + 6\beta n^2 + 12\beta n + 8\beta + \gamma n^2 + 4\gamma n + 4\gamma + \delta n + 2\delta \\
 & - 12\alpha 4^n - 3\beta n^3 - 9\beta n^2 - 9\beta n - 3\beta - 3\gamma n^2 - 6\gamma n - 3\gamma \\
 & - 3\delta n - 3\delta + 2\alpha 4^n + 2\beta n^3 + 2\gamma n^2 + 2\delta n = 4^n + 3n^2
 \end{aligned}$$

$$\begin{cases}
 16\alpha - 12\alpha + 2\alpha = 1 \\
 12\beta + 4\gamma - 8 - 9\beta - 6\gamma - 3\delta + 2\delta = 0 \\
 6\beta + \gamma - 9\beta - 3\gamma + 2\gamma = 3 \\
 8\beta + 4\gamma + 2\delta - 3\beta - 3\gamma - 3\delta = 0
 \end{cases}$$

$$\begin{cases}
 \alpha = 1/6 \\
 \gamma = -3/2 \\
 \beta = -1 \\
 \delta = -13/2
 \end{cases}$$

$$y_{p,n} = \frac{1}{6} 4^n - n^3 - \frac{3}{2} n^2 - \frac{13}{2} n$$

• algemene oplossing:

$$y_n = A \cdot 2^n + B + \frac{4^n}{6} - n^3 - \frac{3n^2}{2} - \frac{13n}{2}$$

i) • homogene vge:

$$\lambda^2 - 6\lambda + 8 = 0$$

$$D = 36 - 32 = 4$$

$$\lambda = \frac{6 \pm 2}{2} = 4 \vee 2$$

$$y_{h,n} = A \cdot 4^n + B \cdot 2^n$$

• 1 particuliere oplossing:

voorstel:  $\alpha \cdot 3^n + \beta n^2 + \gamma n + \delta$

invullen:  $9\alpha 3^n + \beta n^2 + 4\beta n + 4\beta + \gamma n + 2\gamma + \delta - 18\alpha 3^n - 6\beta n^2 - 12\beta n - 6\beta - 6\gamma n - 6\gamma - 6\delta + 8\alpha \cdot 3^n + 8\beta n^2 + 8\gamma n + 8\delta = 2 + 3n^2 - 5 \cdot 3^n$

$$\begin{cases}
 -\alpha = -5 \\
 3\beta = 3 \\
 4\beta + \gamma - 12\beta - 6\gamma + 8\gamma = 0 \\
 4\beta + 2\gamma + \delta - 6\beta - 6\gamma - 6\delta + 8\delta = 2
 \end{cases}
 \quad
 \begin{cases}
 \alpha = 5 \\
 \beta = 1 \\
 \gamma = 8/3 \\
 \delta = \frac{44}{9}
 \end{cases}$$

$$y_{p,n} = 5 \cdot 3^n + n^2 + \frac{8n}{3} + \frac{44}{9}$$

◦ algemene oplossing:

$$y_n = A \cdot 4^n + B \cdot 2^n + 5 \cdot 3^n + n^2 + \frac{8n}{3} + \frac{44}{9}$$

7. a) tweede orde, niet-constante coëfficiënten

b) 2 dimensionale deelruimte van  $\mathcal{R}$

$$c) (n+2)! \lambda^{n+2} + (n+2)! \lambda^{n+1} - 6(n+2)! \lambda^n = 0$$

$$\Leftrightarrow \lambda^2 + \lambda - 6 = 0$$

$$D = 1 + 24 = 25$$

$$\lambda = \frac{-1 \pm 5}{2} = -3 \vee 2$$

$$y_n = A \cdot (-3)^n \cdot n! + B \cdot 2^n \cdot n! \quad \text{met } A, B \in \mathbb{R}$$

Omdat de dimensie van oplossingsruimte gelijk is aan 2, is elk koppel van 2 lineair onafhankelijke rijen van oplossingsruimte een basis voor de oplossingsruimte. We vinden dus alle rijen van oplossingsruimte als lineaire combinatie van die twee rijen.

8. a) 2 en -3 moeten oplossingen zijn van karakteristieke vge:

$$\begin{aligned} (\lambda - 2)(\lambda + 3) &= \lambda^2 - 2\lambda + 3\lambda - 6 \\ &= \lambda^2 + \lambda - 6 \end{aligned}$$

∴ dus de homogene vge is

$$y_{n+2} + y_{n+1} - 6y_n = 0$$

$$\text{Nu is } 6(4)^{n+2} + 6 \cdot 4^{n+1} - 36 \cdot 4^n = \alpha \cdot 4^n$$

$$\alpha = 84$$

$$y_{n+2} + y_{n+1} - 6y_n = 84 \cdot 4^n$$

b) 1. we zoeken een oplossing vte differentieelvge die aan  $\lim_{n \rightarrow +\infty} 2^n \cdot y_n = 5$  voldoet.

$$\text{Nu } y_n = A \cdot \lambda_1^n + B \cdot \lambda_2^n + C$$

$$\text{Dus } 2^n y_n = A \cdot 2\lambda_1^n + B \cdot 2\lambda_2^n + 2^n \cdot C$$

Om als lim. et 5 te hebben moet

$$\lim_{n \rightarrow \infty} A \cdot (2\lambda_1)^n = 0 = \lim_{n \rightarrow \infty} B \cdot (2\lambda_2)^n \text{ en } \lim_{n \rightarrow \infty} C \cdot 2^n = 5$$

$$\text{Dit kan bij } \lambda_1 = \frac{1}{4}, \lambda_2 = \frac{1}{6}, C = 5 \cdot \left(\frac{1}{2}\right)^n$$

De algemere oplossing is dus gegeven door

$$y_n = A \cdot \left(\frac{1}{4}\right)^n + B \cdot \left(\frac{1}{6}\right)^n + 5 \cdot \left(\frac{1}{2}\right)^n$$

De homogene vge die hier bij past is:

$$\left(\lambda - \frac{1}{4}\right)\left(\lambda - \frac{1}{6}\right) = \lambda^2 - \frac{5}{12}\lambda + \frac{1}{24}$$

$$\text{of } y_{n+2} - \frac{5}{12}y_{n+1} + \frac{1}{24}y_n = 0$$

Vullen we nu  $5 \cdot \left(\frac{1}{2}\right)^n$  in en stellen we het gelijk aan  $a \left(\frac{1}{2}\right)^n$  dan vinden we:

$$5 \cdot \left(\frac{1}{2}\right)^{n+2} - \frac{25}{12} \left(\frac{1}{2}\right)^{n+1} + \frac{5}{24} \left(\frac{1}{2}\right)^n = a \left(\frac{1}{2}\right)^n$$

$$a = \frac{5}{12}$$

de lineaire differentieelvge is dan de volgende:

$$y_{n+2} - \frac{5}{12}y_{n+1} + \frac{1}{24}y_n = \frac{5}{12} \left(\frac{1}{2}\right)^n$$

9. • homogene vgl

$$\lambda^2 - \beta\lambda + \beta = 0$$

$$D = \beta^2 - 4\beta$$

①  $\beta = 4 \quad D = 0$

$$(\lambda - 2)^2$$

$$y_n^{(1)} = 2^n$$

$$y_n^{(2)} = n \cdot 2^n$$

$$y_n = A \cdot 2^n + Bn \cdot 2^n$$

②  $\beta < 4 \quad D < 0$

$$\beta \pm \frac{\sqrt{|\beta(\beta - 4)|}}{2} i$$

$$\tilde{y}_n^{(1)} = \frac{\beta + |\beta^2 - 4\beta|^{1/2} i}{2}$$

$$\tilde{y}_n^{(2)} = \frac{\beta - |\beta^2 - 4\beta|^{1/2} i}{2}$$

→ reell machen durch lineare kombinaties

$$y_n^{(1)} = \frac{1}{2} (\tilde{y}_n^{(1)} + \tilde{y}_n^{(2)})$$

$$y_n^{(2)} = \frac{1}{2i} (\tilde{y}_n^{(1)} - \tilde{y}_n^{(2)})$$

$$\text{Nu } r = \sqrt{\frac{\beta^2 + |\beta^2 - 4\beta|}{4}} = \sqrt{\frac{\beta^2 - \beta^2 + 4\beta}{4}} = \sqrt{\beta}$$

$$\cos \theta = \frac{\beta}{2} \frac{1}{\sqrt{\beta}} = \frac{\sqrt{\beta}}{2} \quad \sin \theta = \frac{\sqrt{4\beta - \beta^2}}{2} \cdot \frac{1}{\sqrt{\beta}} \\ = \frac{\sqrt{4 - \beta}}{2}$$

$$y_n^{(1)} = \frac{1}{2} ((\sqrt{\beta})^n (\cos n\theta + i \sin n\theta) + (\sqrt{\beta})^n (\cos n\theta - i \sin n\theta)) \\ = (\sqrt{\beta})^n \cos n\theta \quad \text{met } \cos \theta = \frac{\sqrt{\beta}}{2} \quad \text{en } \sin \theta = \frac{\sqrt{4 - \beta}}{2}$$

$$y_n^{(2)} = \frac{1}{2i} ((\sqrt{\beta})^n (\cos n\theta + i \sin n\theta) - (\sqrt{\beta})^n (\cos n\theta - i \sin n\theta)) \\ = (\sqrt{\beta})^n \sin n\theta \quad \text{met } \cos \theta = \frac{\sqrt{\beta}}{2} \quad \text{en } \sin \theta = \frac{\sqrt{4 - \beta}}{2}$$

$$y_n = (\sqrt{\beta})^n (A \cos n\theta + B \sin n\theta)$$

$$\textcircled{3} \quad \beta > 4 \quad D > 0$$

$$\lambda = \beta \pm \frac{\sqrt{\beta(\beta-4)}}{2}$$

$$y_{n,n} = A \cdot \left( \frac{\beta + \sqrt{\beta(\beta-4)}}{2} \right)^n + B \cdot \left( \frac{\beta - \sqrt{\beta(\beta-4)}}{2} \right)^n$$

• particuliere oplossing

voorstel:  $y$

$$y = \alpha + \beta y - \beta y$$

$$y = \alpha$$

• algemene oplossing:

$$\textcircled{1} \quad \beta = 4 \quad y_n = A \cdot 2^n + B \cdot 2^n + \alpha$$

$$\lim_{n \rightarrow \infty} y_n = +\infty$$

$$\textcircled{2} \quad \beta < 4 \quad y_n = (\sqrt{\beta})^n (A \cos n\theta + B \sin n\theta) + \alpha$$

$$\text{met } \cos \theta = \frac{\sqrt{\beta}}{2} \text{ en } \sin \theta = \frac{\sqrt{4-\beta}}{2}$$

$$\lim_{n \rightarrow \infty} y_n = \alpha \text{ als } \beta < 1$$

$$= +\infty \text{ als } \beta > 1$$

oscilleert rond  $\alpha$  als  $\beta = 1$

$$\textcircled{3} \quad \beta > 4 \quad y_n = A \cdot \left( \frac{\beta + \sqrt{\beta(\beta-4)}}{2} \right)^n + B \cdot \left( \frac{\beta - \sqrt{\beta(\beta-4)}}{2} \right)^n$$

$$\lim_{n \rightarrow \infty} y_n = +\infty$$

## 5.1.4. opdrachten

$$1. \text{ pmvgl: } \begin{cases} x = (1-\lambda)2 + \lambda \cdot 3 \\ y = (1-\lambda)1 + \lambda \cdot (-2) \end{cases} \Leftrightarrow \begin{cases} x = \lambda + 2 \\ y = -3\lambda + 1 \end{cases} \text{ met } \lambda \in \mathbb{R}$$

$$\text{CAvgl: } (y-1)(3-2) = (x-2)(-2-1)$$

$$y-1 = -3x+6$$

$$y = -3x+7$$

$$\text{pmvgl: } \begin{cases} x = (1-\lambda)7 + \lambda \\ y = (1-\lambda)0 + \lambda 4 \end{cases} \begin{cases} x = -6\lambda + 7 \\ y = \lambda 4 \end{cases} \text{ met } \lambda \in \mathbb{R}$$

$$\text{CAvgl: } y = (x-7) \cdot \frac{4}{-6} = -\frac{2}{3}x + \frac{14}{3}$$

$$\text{snijpunt: } \begin{cases} y = -3x+7 \\ y = -\frac{2}{3}x + \frac{14}{3} \end{cases} \begin{cases} y = -3x+7 \\ -3x+7 = -\frac{2}{3}x + \frac{14}{3} \end{cases}$$

$$\begin{cases} y = 4 \\ x = 1 \end{cases} \rightarrow \text{snijpunt } (1,4)$$

$$2. \text{ manier 1: } \frac{|3 \cdot 8 + 2 \cdot 5 - 8|}{\sqrt{3^2 + 2^2}} = 2\sqrt{13}$$

manier 2: loodrecht op rechte is volgens richting  
 $(a,b) \rightarrow (3,2)$

$$\text{pmvgl rechte door } (8,5) \begin{cases} x = 8 + 3\lambda \\ y = 5 + 2\lambda \end{cases}$$

$$\text{met richting } (3,2)$$

$$\text{CAvgl: } 3y - 2x + 1 = 0$$

$$\text{snijpunt: } \begin{cases} 3y - 2x + 1 = 0 \\ 3x + 2y - 8 = 0 \end{cases} \begin{cases} y = 1 \\ x = 2 \end{cases}$$

$$\text{afstand tss } (2,1) \text{ en } (8,5) = \sqrt{(8-2)^2 + (5-1)^2} = 2\sqrt{13}$$

3. Stel  $x$  = aantal eenheden  $V_1$

$y$  = aantal eenheden  $V_2$

$$W: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, y) \rightarrow 15x + 9y$$

De randvoorwaarden zijn:

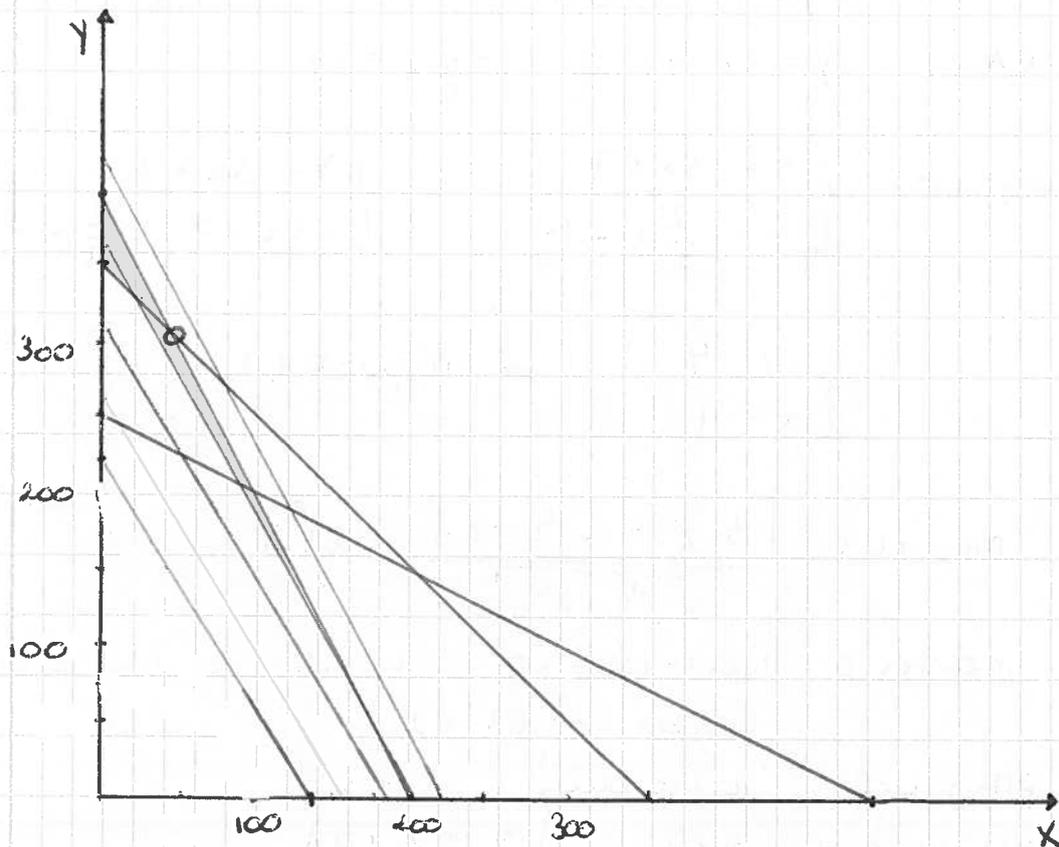
$$2x + y \geq 400$$

$$x + 2y \geq 500$$

$$4x + 4y \geq 1400$$

$$x \geq 0$$

$$y \geq 0$$



Snijpunt tussen  $2x + y = 400$  en  $x + 2y = 350$

$$\begin{cases} 700 - 2y + y = 400 \\ x = 350 - y \end{cases} \quad \begin{cases} y = 300 \\ x = 50 \end{cases}$$

$$W(50, 300) = 15 \cdot 50 + 9 \cdot 300 = 3450 = \text{€ } 34,50$$

4.  $x =$  aantal HDT

$y =$  aantal LDT

$$W: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, y) \rightarrow 400x + 240y - 40 \cdot 4,8x - 20 \cdot 4,8y - x(20 \cdot 4 + 40 \cdot 1,6) - y(40 \cdot 1,6)$$

$$W: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, y) \rightarrow 64x + 80y$$

randvoorwaarden:

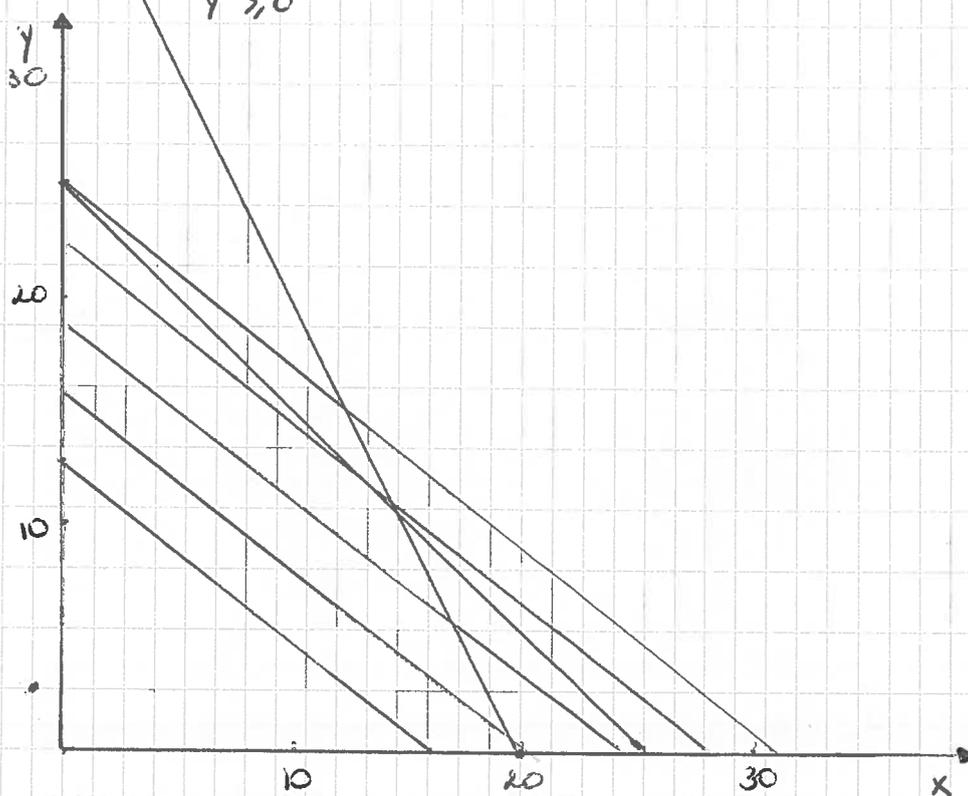
$$20x \leq 1200$$

$$40x + 40y \leq 1000$$

$$40x + 20y \leq 800$$

$$x \geq 0$$

$$y \geq 0$$



Snijpunt  $40x + 40y \leq 1000$  en  $x = 0$

$$\begin{cases} x + y = 25 \\ x = 0 \end{cases} \quad \begin{cases} y = 25 \\ x = 0 \end{cases}$$

$$W(0, 25) = 80 \cdot 25 = 2000$$

5.  $x =$  aantal pakket 1  
 $y =$  aantal pakket 2

$$W: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, y) \rightarrow 15x + 20y$$

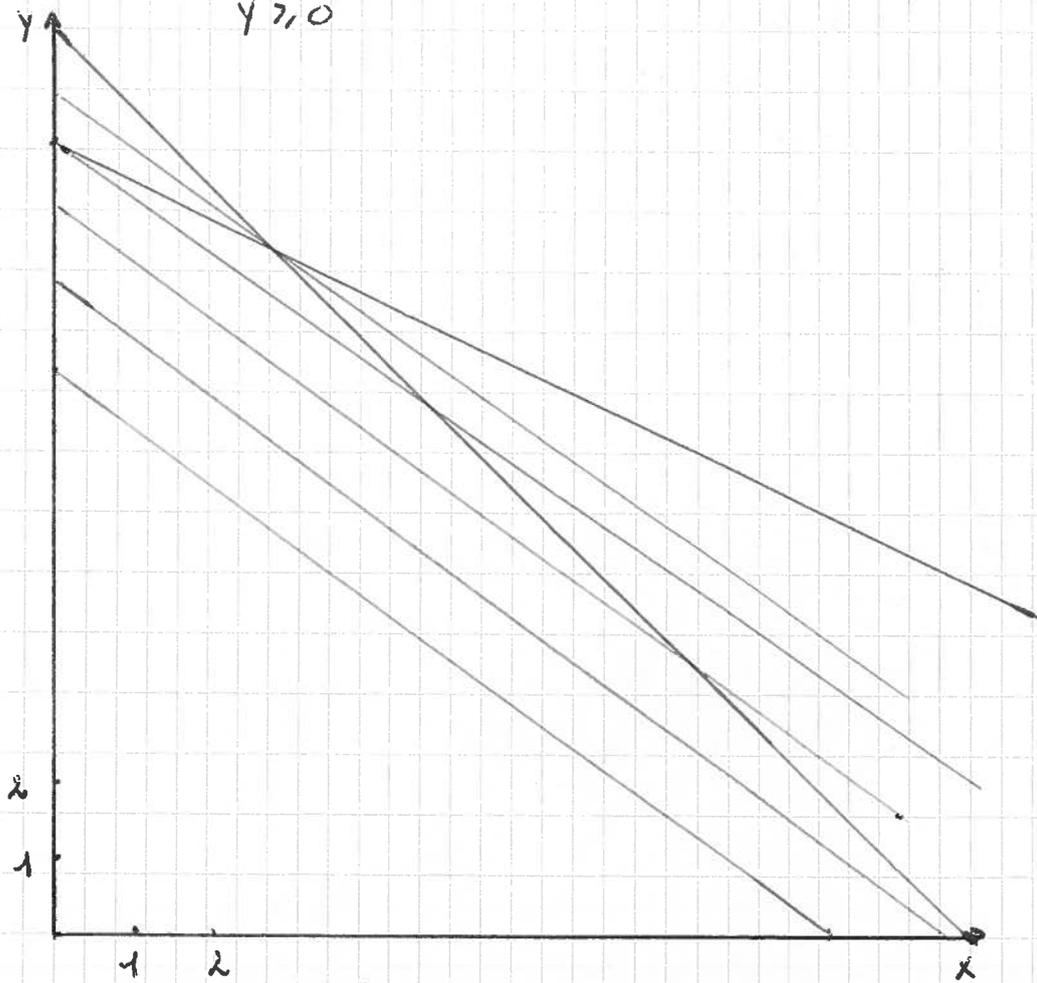
randvoorwaarden:

$$4x + 8y \leq 84$$

$$2x + 2y \leq 24$$

$$x \geq 0$$

$$y \geq 0$$



Snijpunt tussen  $4x + 8y = 84$  en  $2x + 2y = 24$

$$\begin{cases} 4x + 8y = 84 \\ x + y = 12 \end{cases}$$

$$\begin{cases} 4(12 - y) + 8y = 84 \\ x = 12 - y \end{cases}$$

$$\begin{cases} y = 9 \\ x = 3 \end{cases}$$

$$W(3, 9) = 15 \cdot 3 + 20 \cdot 9 = 225$$

### 5.2.3. opdrachten

$$1. \det \begin{pmatrix} a_1 & b_1 & p_1 - x \\ a_2 & b_2 & p_2 - y \\ a_3 & b_3 & p_3 - z \end{pmatrix} = 0$$

$$a_1(b_2(z-p_3) - b_3(y-p_2)) - b_1(a_2(z-p_3) - a_3(y-p_2)) + (x-p_1)(a_2b_3 - a_3b_2) = 0$$

$$\Leftrightarrow (a_2b_3 - a_3b_2)x + (a_3b_1 - b_3a_1)y + (b_2a_1 - b_1a_2)z - a_1b_2p_3 + a_1b_3p_2 + b_1a_2p_3 - b_1a_3p_2 - a_2b_3p_1 + p_1a_3b_2 = 0$$

$$a = a_2b_3 - a_3b_2$$

$$b = a_3b_1 - b_3a_1$$

$$c = b_2a_1 - b_1a_2$$

$$d = a_1b_2p_3 - a_1b_3p_2 - b_1a_2p_3 + b_1a_3p_2 + a_2b_3p_1 - p_1a_3b_2$$

$$2. a) \begin{cases} x = (1 - \lambda_1 - \lambda_2) + 2\lambda_1 + 4\lambda_2 \\ y = (1 - \lambda_1 - \lambda_2) \cdot 2 + 3\lambda_1 - \lambda_2 \\ z = (1 - \lambda_1 - \lambda_2) \cdot (-4) + 7\lambda_1 + 3\lambda_2 \end{cases}$$

$$\begin{cases} x = 1 + \lambda_1 + 3\lambda_2 \\ y = 2 + \lambda_1 - 3\lambda_2 \\ z = -4 + 11\lambda_1 + 7\lambda_2 \end{cases} \quad \begin{cases} \lambda_1 = x - 1 - 3\lambda_2 \\ y = 2 + x - 1 - 6\lambda_2 \\ z = -4 + 11\lambda_1 + 7\lambda_2 \end{cases}$$

$$\begin{cases} \dots \\ \lambda_2 = (1 + x - y) / 6 \\ z = -4 + 11x - 11 - 5,5 - 5,5x + 5,5y + \frac{7}{6} + \frac{7}{6}x - \frac{7}{6}y \end{cases}$$

$$\begin{cases} \dots \\ \dots \\ 6z = -116 + 40x + 26y \end{cases} \quad 20x + 13y - 3y = 58$$

$$b) \begin{cases} x = (1 - \lambda_1 - \lambda_2)(-7) + 2\lambda_1 + 4\lambda_2 \\ y = (1 - \lambda_1 - \lambda_2) - \lambda_1 + \lambda_2 \\ z = (1 - \lambda_1 - \lambda_2) \cdot 0 + 3\lambda_1 + 6\lambda_2 \\ x = -7 - 4,5(y-1) + (11/6)(z + 1,5(y-1)) \\ \lambda_1 = (y-1)/(1-2) \\ \lambda_2 = (z - 3\lambda_1)/6 \end{cases}$$

$$x = -7 - 4,5y + 4,5 + \frac{11z}{6} + \frac{11}{4}y - \frac{11}{4}$$

$$12x = -63 - 21y + 22z$$

$$-12x - 21y + 22z = -63$$

$$3. a) \begin{cases} x = (1-\lambda)2 + 2\lambda \\ y = (1-\lambda)3 + 0\lambda \\ z = (1-\lambda)(-4) - 4\lambda \end{cases} \quad \begin{cases} x = 2 \\ y = 3 - 3\lambda \\ z = -4 \end{cases} \quad \text{PMVgl}$$

$$\text{CAVgl} \begin{cases} x = 2 \\ z = -4 \end{cases}$$

$$\text{Vektorrechte } v = (2, 3, -4) + \lambda(0, -3, 0)$$

$$b) \begin{cases} x = (1-\lambda)2 + \lambda \\ y = (1-\lambda) + 2\lambda \\ z = (1-\lambda)3 - \lambda \end{cases} \quad \begin{cases} x = 2 - \lambda \\ y = 1 + \lambda \\ z = 3 - 4\lambda \end{cases} \quad \text{PMVgl}$$

$$\text{CAVgl} \begin{cases} x = 2 - (y-1) \\ z = 3 - 4(2-x) \end{cases} \quad \begin{cases} x = 3 - y \\ z = -5 + 4x \end{cases}$$

$$\text{Vektorrechte : } (2, 1, 3) + \lambda(-1, 1, -4)$$

$$\begin{cases} 2x+2=3y+9 & \rightarrow \text{richting } x = \frac{3}{2}y \\ 2z-4 = -4y-12 & \rightarrow \text{richting } z = -2y \end{cases}$$

$$\begin{cases} x = \frac{3}{2}y + \alpha \\ z = -2y + \beta \end{cases} \quad \begin{cases} 3 = \frac{3}{2} + \alpha \\ -2 = -2 + \beta \end{cases} \quad \begin{cases} \alpha = \frac{3}{2} \\ \beta = 0 \end{cases}$$

$$\text{CA vgl } \begin{cases} 2x = 3y + 3 \\ z = -2y \end{cases}$$

5. uit vgl 2 volgt

$$z = 2\lambda$$

$$y = \lambda$$

en uit vgl 1:

$$x = 2\lambda - 1$$

$$\text{PM vgl: } \begin{cases} x = 2\lambda \\ y = \lambda \\ x = 2\lambda - 1 \end{cases} \quad \lambda \in \mathbb{R}$$

$$\sqrt{(2\lambda - 1 - 1)^2 + (\lambda - 1)^2 + (2\lambda - 2)^2} = 6$$

$$\Leftrightarrow 4\lambda^2 + 4 - 8\lambda + \lambda^2 - 2\lambda + 1 + 4\lambda^2 + 4 - 8\lambda = 36$$

$$9\lambda^2 - 18\lambda - 27 = 0$$

$$D = 324 + 972 = 1296 = 36^2$$

$$\frac{18 \pm 36}{18} \begin{matrix} / 3 \\ \setminus -1 \end{matrix}$$

De punten zijn dus  $(5, 3, 6)$  en  $(-3, -1, -2)$

$$6. \text{ afstand} = \frac{|10 + 18 + 2 - 6|}{\sqrt{4 + 9 + 1}} = \frac{12\sqrt{14}}{7}$$

$$7. \begin{cases} x = (1-\lambda)3 + \lambda \\ y = (1-\lambda)2 + 5\lambda \\ z = (1-\lambda)(-3) + 0 \cdot \lambda \end{cases} \quad \begin{cases} x = 3 - 2\lambda \\ y = 2 + 3\lambda \\ z = -3 + 3\lambda \end{cases} \quad \lambda \in \mathbb{R}$$

$$\begin{cases} y = 2 + 3\left(\frac{x-3}{-2}\right) \\ z = -3 + 3\left(\frac{x-3}{-2}\right) \end{cases} \quad \begin{cases} y = 6,5 - 1,5x \\ z = -1,5 - 1,5x \end{cases}$$

Om de afstand te berekenen, construeren we een vlak loodrecht op de rechte die p bevat.

Het vlak heeft de volgende vergelijking:

$$-2x + 3y + 3z + k = 0. \text{ Vullen we } p \text{ in vinden we voor } k: -16 + 12 + k = 0 \Leftrightarrow k = 4$$

Nu zoeken we  $\lambda$  zodat het punt

$(3-2\lambda, 2+3\lambda, -3+3\lambda)$  in het vlak behoort.

$$\Rightarrow -6 + 4\lambda + 6 + 9\lambda - 9 + 9\lambda + 4 = 0$$

$$\lambda = \frac{5}{22}$$

Het punt op loodrechte afstand van p op de rechte is dus  $(\frac{28}{11}, \frac{59}{22}, \frac{-51}{22})$

De afstand tussen deze twee punten is:

$$\sqrt{(28/11 - 8)^2 + (59/22)^2 + (-51/22 - 4)^2}$$

$$= \sqrt{\left(\frac{-60}{11}\right)^2 + \left(\frac{59}{22}\right)^2 + \left(\frac{-139}{22}\right)^2}$$

$$= \sqrt{\frac{14400 + 3481 + 19321}{484}}$$

$$= \frac{\sqrt{37202}}{22}$$

### 6.3 opdrachten

$$1. a) \det \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} = -\lambda(1-\lambda)^2$$

eigenwaarden zijn 0, 1

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} y = 0 \\ x + z = 0 \end{cases} \Rightarrow \text{eigenvector } (a, 0, -a) \mid a \in \mathbb{R}_0$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x = 0 \\ x = 0 \end{cases} \Rightarrow \text{eigenvector } \{(0, r, s) \mid r, s \in \mathbb{R} \\ \text{en } (r, s) \neq (0, 0)\}$$

diagonaliseerbaar met basis van  $\mathbb{R}^3$

$$\{(1, 0, -1), (0, 0, 1), (0, 1, 0)\}$$

$$b) \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^3$$

eigenwaarden zijn 1

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\{x = 0 \Rightarrow \text{eigenvector } \{(0, a, b) \mid a, b \in \mathbb{R} \text{ met } (a, b) \neq (0, 0)\}$$

→ niet diagonaliseerbaar want geen basis van  $\mathbb{R}^3$

$$c) \det \begin{pmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{pmatrix} = (-3-\lambda) \left( (5-\lambda)(-2-\lambda) + 6 \right) - \left( (14+7\lambda) - 6 \right) - (-4\lambda + 30 - 6\lambda) = 0$$

$$\Leftrightarrow (-3-\lambda)(\lambda^2 - 3\lambda - 4) - 8 - 7\lambda + 12 + 6\lambda = 0$$

$$\Leftrightarrow -\lambda^3 - 3\lambda^2 + 3\lambda^2 + 9\lambda + 4\lambda + 12 + 4 - \lambda = 0$$

$$\Leftrightarrow -\lambda^3 + 12\lambda + 16 = 0$$

$$\begin{array}{c|ccc} -1 & 0 & 12 & 16 \\ -2 & 2 & -4 & -16 \\ \hline -1 & 2 & 8 & 0 \end{array}$$

$$(\lambda + 2)(-\lambda^2 + 2\lambda + 8) = 0$$

$$D = 4 + 32$$

$$\frac{-2 \pm 6}{-2} \begin{matrix} / 4 \\ / -2 \end{matrix}$$

$$(\lambda + 2)(\lambda + 2)(\lambda - 4) = 0$$

eigenwaarden zijn -2 en 4

$$\begin{pmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x + y - z = 0 \\ -7x + 7y - z = 0 \\ -6x + 6y = 0 \end{cases} \begin{cases} z = 0 \\ x = y \\ x = y \end{cases}$$

→ eigenvector  $\{(a, a, 0) \mid a \in \mathbb{R}_0\}$

$$\begin{pmatrix} -7 & 1 & -1 \\ -7 & 1 & -1 \\ -6 & 6 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -7x + y - z = 0 \\ -7x + y - z = 0 \\ -6x + 6y - 6z = 0 \end{cases} \quad \begin{cases} \dots \\ -7y + 7z + y - z = 0 \\ x = y - z \end{cases}$$

$$\begin{cases} y = z \\ x = y - z \end{cases} \quad \begin{cases} y = z \\ x = 0 \end{cases} \rightarrow \text{eigenvector } \{(0, a, a) \mid a \in \mathbb{R}_0\}$$

$\rightarrow$  niet diagonaliseerbaar want geen basis van  $\mathbb{R}^3$

$$\begin{aligned} 2. \lim_{k \rightarrow \infty} A^k &= \lim_{k \rightarrow \infty} (Q D Q^{-1})^k = \lim_{k \rightarrow \infty} Q \begin{pmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_n^k \end{pmatrix} Q^{-1} \\ &= Q \cdot 0_{m \times m} \cdot Q^{-1} = 0_{m \times m} \end{aligned}$$

$$\begin{aligned} 3. a) \det \begin{pmatrix} 0,25 - \lambda & 0 & 0 \\ 0,50 & 0,18 - \lambda & 0 \\ 0 & 0,75 & 0,02 - \lambda \end{pmatrix} &= 0 \\ \Leftrightarrow (0,25 - \lambda)(0,18 - \lambda)(0,02 - \lambda) &= 0 \\ \Leftrightarrow (0,25 - \lambda)(\lambda^2 - 0,2\lambda + 0,0036) &= 0 \end{aligned}$$

eigenwaarden:  $0,25; 0,18; 0,02$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0,5 & -0,07 & 0 \\ 0 & 0,75 & -0,23 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 0,5x - 0,07y = 0 \\ 0,75y - 0,23z = 0 \end{cases} \quad \begin{cases} x = 0,14y \\ z = 3,26y \end{cases}$$

$\rightarrow$  eigenvector  $\{(0,14r; r; 3,26r) \mid r \in \mathbb{R}_0\}$

$$\begin{pmatrix} 0,07 & 0 & 0 \\ 0,5 & 0 & 0 \\ 0 & 0,75 & -0,16 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x=0 \\ y=0,21z \end{cases} \rightarrow \text{eigenvector } \{(0; 0,21a; a) \mid a \in \mathbb{R}_0\}$$

$$\begin{pmatrix} 0,23 & 0 & 0 \\ 0,5 & 0,16 & 0 \\ 0 & 0,75 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x=0 \\ y=0 \end{cases} \rightarrow \text{eigenvector } \{(0, 0, a) \mid a \in \mathbb{R}_0\}$$

b) Omdat alle eigenwaarden kleiner zijn dan nul, zal  $\lim_{k \rightarrow \infty} A^k = 0$ .  $x_n$  wordt dan:

$$\lim_{n \rightarrow \infty} x_n = (\mathbb{1}_3 - A)^{-1} \cdot b$$

$$= \begin{pmatrix} 0,75 & 0 & 0 \\ -0,5 & 0,82 & 0 \\ 0 & -0,75 & 0,98 \end{pmatrix}^{-1} \begin{pmatrix} 200 \\ 0 \\ 0 \end{pmatrix}$$

$$\det \begin{pmatrix} 0,75 & 0 & 0 \\ -0,5 & 0,82 & 0 \\ 0 & -0,75 & 0,98 \end{pmatrix} = 0,6027$$

$$(\mathbb{1}_3 - A)^{-1} = \frac{1}{0,6027} \begin{pmatrix} 0,8036 & \cdot & \cdot \\ 0,49 & \cdot & \cdot \\ 0,375 & \cdot & \cdot \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1,3333 & \cdot & \cdot \\ 0,8130 & \cdot & \cdot \\ 0,6222 & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 200 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 267 \\ 163 \\ 124 \end{pmatrix}$$

$$4. (x_1, \dots, x_n) A = \lambda (x_1, \dots, x_n)$$

Want: we weten dat

$Ax = \lambda x$ . Transponeren we dit krijgen we

$$x^T \cdot A^T = \lambda \cdot x^T$$

$$x^T \cdot A = \lambda \cdot x^T$$

5. ① Als  $\lambda$  een eigenwaarde is van  $A$  is ze dat ook van  $B$

$$Ax = \lambda x \quad \text{met } x \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Leftrightarrow C^{-1} B C x = \lambda x$$

$$\Leftrightarrow B C x = C \lambda x$$

$$\Leftrightarrow B C x = \lambda C x$$

Nu tonen we nog aan dat  $Cx \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

veronderstel dat  $Cx = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

$$\Leftrightarrow C^{-1} C x = C^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$x = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\rightarrow$  tegenspraak met geg.

② Als  $\lambda$  een eigenwaarde is van  $B$  is ze dat ook van  $A$

$$Bx = \lambda x \quad \text{met } x \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Leftrightarrow C A C^{-1} x = \lambda x$$

$$\Leftrightarrow C^{-1} C A C^{-1} x = C^{-1} \lambda x$$

$$\Leftrightarrow A C^{-1} x = \lambda C^{-1} x$$

Nu tonen we nog aan dat  $C^{-1}x \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

veronderstel dat  $C^{-1}x = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

$$\Leftrightarrow C C^{-1} x = C \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$x = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\rightarrow$  tegenspraak geg.

eigenvector  $A = C^{-1}$  eigenvector  $B$

6. Beschouw een  $m \times m$  bovendriehoeksmatrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ 0 & a_{22} & \dots & a_{2m} \\ 0 & 0 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{mm} \end{bmatrix}$$

We zoeken de eigenwaarden van deze matrix.

Hiervoor berekenen we de determinant van volgende matrix

$$\det \begin{bmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1m} \\ 0 & a_{22}-\lambda & \dots & \vdots \\ 0 & 0 & \dots & \vdots \\ 0 & 0 & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & a_{mm}-\lambda \end{bmatrix}$$

ontwikkelen naar de eerste kolom geeft

$$(a_{11}-\lambda) \det \begin{bmatrix} a_{22}-\lambda & \dots & a_{2m} \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_{mm}-\lambda \end{bmatrix}$$

nogmaals ontwikkelen naar de eerste kolom geeft:

$$(a_{11}-\lambda)(a_{22}-\lambda) \det \begin{bmatrix} a_{33}-\lambda & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_{mm}-\lambda \end{bmatrix}$$

zo blijf je verder doen. uiteindelijk krijgen we

$$(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) \dots (a_{mm}-\lambda)$$

of nog de elementen vld diagonaal zijn de eigenwaarden.

7. a) geg:  $e_1 = \frac{v_1}{\|v_1\|}$        $\tilde{v}_2 = v_2 - \langle e_1, v_2 \rangle e_1$

TB:  $\langle e_1, \tilde{v}_2 \rangle = 0$

Bewijs:

$$\begin{aligned}
 \langle e_1, \tilde{v}_2 \rangle &= \left\langle \frac{v_1}{\|v_1\|}, v_2 - \left\langle \frac{v_1}{\|v_1\|}, v_2 \right\rangle \frac{v_1}{\|v_1\|} \right\rangle \\
 &= \frac{v_{1,1} \cdot \left( v_{2,1} - \left\langle v_1, v_2 \right\rangle \frac{v_{1,1}}{\|v_1\|^2} \right) + \dots + v_{1,m} \cdot \left( v_{2,m} - \left\langle v_1, v_2 \right\rangle \frac{v_{1,m}}{\|v_1\|^2} \right)}{\|v_1\|} \\
 &= \frac{\|v_1\|^2 \cdot v_{1,1} \cdot v_{2,1} - \left\langle v_1, v_2 \right\rangle v_{1,1}^2 + \dots + \|v_1\|^2 v_{1,m} v_{2,m} - \left\langle v_1, v_2 \right\rangle v_{1,m}^2}{\|v_1\|^3} \\
 &= \frac{\|v_1\|^2 (v_{1,1} v_{2,1} + \dots + v_{1,m} v_{2,m}) - \left\langle v_1, v_2 \right\rangle (v_{1,1}^2 + v_{1,2}^2 + \dots + v_{1,m}^2)}{\|v_1\|^3} \\
 &= \frac{\|v_1\|^2 \left\langle v_1, v_2 \right\rangle - \left\langle v_1, v_2 \right\rangle \|v_1\|^2}{\|v_1\|^3} \\
 &= 0
 \end{aligned}$$

TB:  $\tilde{v}_2 \neq 0$

Bewijs: veronderstel dat  $\tilde{v}_2 = 0$ .

Dan geldt:  $v_2 - \left\langle e_1, v_2 \right\rangle e_1 = 0$

$$v_2 = \frac{\left\langle e_1, v_2 \right\rangle}{\|v_1\|} \cdot v_1$$

Dit is tegenspraak met het gegeven dat  $\{v_1, v_2, \dots\}$  een basis is van  $V$  want dan is  $\{v_1, v_2\}$  vrij.

b) TB:  $\langle e_i, \tilde{v}_3 \rangle = 0$  voor  $i = 1, 2$

$$\begin{aligned}
 \text{Bewijs } \langle e_1, \tilde{v}_3 \rangle &= \left\langle \frac{v_1}{\|v_1\|}, v_3 - \left\langle \frac{v_1}{\|v_1\|}, v_3 \right\rangle \frac{v_1}{\|v_1\|} - \left\langle \frac{\tilde{v}_2}{\|\tilde{v}_2\|}, v_3 \right\rangle \frac{\tilde{v}_2}{\|\tilde{v}_2\|} \right\rangle \\
 &= \frac{v_{1,1} \cdot \left( v_{3,1} - \left\langle v_1, v_3 \right\rangle \frac{v_{1,1}}{\|v_1\|^2} - \left\langle \tilde{v}_2, v_3 \right\rangle \frac{\tilde{v}_{2,1}}{\|\tilde{v}_2\|^2} \right) + \dots}{\|v_1\|^2} \\
 &= \frac{\left\langle v_1, v_3 \right\rangle - \left\langle v_1, v_3 \right\rangle \frac{\|v_1\|^2}{\|v_1\|^2} - \left\langle \tilde{v}_2, v_3 \right\rangle \frac{\left\langle v_1, \tilde{v}_2 \right\rangle}{\|\tilde{v}_2\|^2}}{\|v_1\|^2}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{-\langle \tilde{v}_2, v_3 \rangle \langle v_1, \tilde{v}_2 \rangle}{\|v_1\|^2 \|\tilde{v}_2\|^2} \\
&= \frac{-\langle v_2 - \langle e_1, v_2 \rangle e_1, v_3 \rangle \langle v_1, v_2 - \langle e_1, v_2 \rangle e_1 \rangle}{\|v_1\|^2 \|\tilde{v}_2\|^2} \\
&= \frac{-\left( v_{3,1} (v_{2,1} - \langle v_1, v_2 \rangle \frac{v_{1,1}}{\|v_1\|^2}) + \dots \right) \left( v_{1,1} (v_{2,1} - \langle v_1, v_2 \rangle \frac{v_{1,1}}{\|v_1\|^2}) + \dots \right)}{\|v_1\|^2 \|\tilde{v}_2\|^2} \\
&= \frac{-\left( \langle v_3, v_2 \rangle - \langle v_1, v_2 \rangle \frac{\langle v_3, v_1 \rangle}{\|v_1\|^2} \right) \left( \langle v_1, v_2 \rangle - \langle v_1, v_2 \rangle \cdot \frac{\|v_1\|^2}{\|v_1\|^2} \right)}{\|v_1\|^2 \|\tilde{v}_2\|^2} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\langle e_2, \tilde{v}_3 \rangle &= \left\langle \frac{\tilde{v}_2}{\|\tilde{v}_2\|}, v_3 - \frac{\langle v_1, v_3 \rangle v_1}{\|v_1\|^2} - \frac{\langle \tilde{v}_2, v_3 \rangle \tilde{v}_2}{\|\tilde{v}_2\|^2} \right\rangle \\
&= \frac{\langle v_2 - \langle v_1, v_2 \rangle \frac{v_1}{\|v_1\|}, v_3 - \langle v_1, v_3 \rangle \frac{v_1}{\|v_1\|^2} - \langle \tilde{v}_2, v_3 \rangle \frac{\tilde{v}_2}{\|\tilde{v}_2\|^2} \rangle}{\|\tilde{v}_2\|} \\
&= \frac{\langle v_2, v_3 \rangle - \frac{\langle v_1, v_2 \rangle \langle v_1, v_3 \rangle}{\|v_1\|^2} - \frac{\langle v_1, v_3 \rangle \langle v_1, v_2 \rangle}{\|v_1\|^2} + \frac{\langle v_1, v_2 \rangle \langle v_1, v_3 \rangle}{\|v_1\|^2}}{\|\tilde{v}_2\|} \\
&= \frac{-\frac{\langle \tilde{v}_2, v_3 \rangle \langle v_2, \tilde{v}_2 \rangle}{\|\tilde{v}_2\|^2} + \frac{\langle v_1, v_2 \rangle \langle \tilde{v}_2, v_3 \rangle \langle v_1, \tilde{v}_2 \rangle}{\|v_1\|^2 \|\tilde{v}_2\|^2}}{\|\tilde{v}_2\|}
\end{aligned}$$

$$\begin{aligned}
&e_1, \tilde{v}_2 \rangle = 0 \\
&\langle v_1, \tilde{v}_2 \rangle = 0 \\
&= \frac{\langle v_3, v_2 - \langle v_1, v_2 \rangle \frac{v_1}{\|v_1\|^2} \rangle - \langle v_2, \langle \tilde{v}_2, v_3 \rangle \frac{\tilde{v}_2}{\|\tilde{v}_2\|^2} \rangle + \frac{\langle v_1, v_2 \rangle v_1}{\|v_1\|^2} \frac{\langle \tilde{v}_2, v_3 \rangle \tilde{v}_2}{\|\tilde{v}_2\|^2} \rangle}{\|\tilde{v}_2\|}
\end{aligned}$$

$$= \frac{\langle v_3, \tilde{v}_2 \rangle - \frac{\langle \tilde{v}_2, v_3 \rangle \langle \tilde{v}_2, \tilde{v}_2 \rangle}{\|\tilde{v}_2\|^2}}{\|\tilde{v}_2\|}$$

$$= \frac{\langle v_3, \tilde{v}_2 \rangle - \langle \tilde{v}_2, v_3 \rangle}{\|\tilde{v}_2\|}$$

$$= 0$$

TB:  $\tilde{v}_3 \neq 0$

Bewijs stel dat  $\tilde{v}_3 = 0$ . Dan geldt:

$$v_3 = \langle e_1, v_3 \rangle e_1 + \langle e_2, v_3 \rangle e_2$$

$$v_3 = \langle e_1, v_3 \rangle \frac{v_1}{\|v_1\|} + \langle e_2, v_3 \rangle \frac{\tilde{v}_2}{\|\tilde{v}_2\|}$$

$$v_3 = \langle e_1, v_3 \rangle \frac{v_1}{\|v_1\|} + \frac{\langle e_2, v_3 \rangle}{\|\tilde{v}_2\|} \left( v_2 - \frac{\langle e_1, v_1 \rangle v_1}{\|v_1\|} \right)$$

$v_3$  is dus te schrijven als lineaire combinatie van  $v_1, v_2$  wat niet kan volgens het gegeven dat  $\{v_1, v_2, v_3, \dots, v_n\}$  vrij is.

c)

$$8. A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Deze matrix is duidelijk stochastisch  
maar niet alle elementen zijn  $> 0$ .

Bij eigenwaarde 1 hoort volgende eigenvector

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} y = 0 \\ -y = 0 \end{cases} \Rightarrow \text{eigenvector } \{(a, 0) \mid a \in \mathbb{R}_0\}$$

De eigenvector bevat dus een nul.

$$9. P = \begin{pmatrix} 1/2 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}$$

Volgende karakteristieke vgl hoort hierbij

$$\det \begin{pmatrix} 1/2 - \lambda & 1/4 & 1/2 \\ 1/4 & 1/4 - \lambda & 1/4 \\ 1/4 & 1/2 & 1/4 - \lambda \end{pmatrix} = 0$$

$$\Leftrightarrow \left(\frac{1}{2} - \lambda\right) \left( \left(\frac{1}{4} - \lambda\right)^2 - \frac{1}{8} \right) - \frac{1}{4} \left( \frac{1}{4} \left(\frac{1}{4} - \lambda\right) - \frac{1}{16} \right) + \frac{1}{2} \left( \frac{1}{8} - \frac{1}{4} \left(\frac{1}{4} - \lambda\right) \right) = 0$$

$$\Leftrightarrow \frac{1}{32} + \frac{\lambda^2}{2} - \frac{1}{4}\lambda - \frac{1}{16}\lambda - \lambda^3 + \frac{\lambda^2}{2} - \frac{1}{16} + \frac{\lambda}{8} - \frac{1}{64} + \frac{\lambda}{16} + \frac{1}{64} + \frac{1}{16} - \frac{1}{32} + \frac{\lambda}{8} = 0$$

$$\Leftrightarrow 1 + 16\lambda^2 - 8\lambda - 2\lambda - 32\lambda^3 + 16\lambda^2 - 2 + 4\lambda + 2\lambda + 2 - 1 + 4\lambda = 0$$

$$\Leftrightarrow -32\lambda^3 + 32\lambda^2 = 0$$

$$\lambda = 1 \vee \lambda = 0$$

$$\begin{pmatrix} -1/2 & 1/4 & 1/2 \\ 1/4 & -3/4 & 1/4 \\ 1/4 & 1/2 & -3/4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x + y + 2z = 0 \\ x - 3y + z = 0 \\ x + 2y - 3z = 0 \end{cases} \quad \begin{cases} -6z + 4y + y + 2z = 0 \\ 3z - 2y - 3y + z = 0 \\ x = 3z - 2y \end{cases}$$

$$\begin{cases} -\frac{30}{4}y + 4y + y + \frac{5}{2}y = 0 \\ z = \frac{5}{4}y \\ x = 3z - 2y \end{cases} \quad \begin{cases} 0 = 0 \\ z = \frac{5}{4}y \\ x = \frac{7}{4}y \end{cases}$$

→ eigenvector  $\left\{ \left( \frac{7}{4}a, a, \frac{5}{4}a \right) \mid a \in \mathbb{R}_0 \right\}$

$$\begin{pmatrix} 1/2 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x + y + 2z = 0 \\ x + y + z = 0 \\ x + 2y + z = 0 \end{cases} \quad \begin{cases} 0 = 0 \\ -y = 0 \\ x = -2y - z \end{cases} \quad \begin{cases} y = 0 \\ 0 = 0 \\ x = -z \end{cases}$$

→ eigenvector  $\left\{ (a, 0, -a) \mid a \in \mathbb{R}_0 \right\}$

b)  $P$  is niet diagonaliseerbaar omdat men geen drie lineair onafhankelijke vectoren kan maken uit de eigenvectoren. Hierdoor kunnen we geen basis van  $\mathbb{R}^3$  maken.

c) we zoeken  $a$  zodat  $\frac{7a}{4} + a + \frac{5a}{4} = 1$

$$\Leftrightarrow \frac{16a}{4} = 1$$

$$\Leftrightarrow a = \frac{1}{4}$$

$$x^* = \left( \frac{7}{16}, \frac{1}{4}, \frac{5}{16} \right)$$

$$\begin{pmatrix} 1/2 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}^2 = \begin{pmatrix} 7/16 & 7/16 & 7/16 \\ 1/4 & 1/4 & 1/4 \\ 5/16 & 5/16 & 5/16 \end{pmatrix}$$

$$\begin{pmatrix} 7/16 & 7/16 & 7/16 \\ 1/4 & 1/4 & 1/4 \\ 5/16 & 5/16 & 5/16 \end{pmatrix} \begin{pmatrix} 1/2 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \end{pmatrix} = \begin{pmatrix} 7/16 & 7/16 & 7/16 \\ 1/4 & 1/4 & 1/4 \\ 5/16 & 5/16 & 5/16 \end{pmatrix}$$

Dus  $\lim_{n \rightarrow \infty} P^n = P^*$

Nu is  $\lim_{n \rightarrow \infty} P^n \cdot x = \begin{pmatrix} 7/16 & 7/16 & 7/16 \\ 1/4 & 1/4 & 1/4 \\ 5/16 & 5/16 & 5/16 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$  met  $x_1 + x_2 + x_3 = 1$

$$= \begin{pmatrix} 7/16 (x_1 + x_2 + x_3) \\ 1/4 (x_1 + x_2 + x_3) \\ 5/16 (x_1 + x_2 + x_3) \end{pmatrix}$$

$$= \begin{pmatrix} 7/16 \\ 1/4 \\ 5/16 \end{pmatrix}$$

10. We berekenen eerst de eigenwaarden en eigenvectoren van de matrix.

$$\det \begin{pmatrix} 3-\lambda & -1 & 1 \\ -1 & 3-\lambda & -5 \\ 0 & 0 & -2-\lambda \end{pmatrix} = (-2-\lambda) \left( (3-\lambda)^2 - 1 \right) = 0$$

$$= (-2-\lambda) (\lambda^2 - 6\lambda + 8) = 0$$

$$D = 36 - 32 = 4$$

$$\lambda = -2 \vee \lambda = \frac{6 \pm 2}{2} = 4, 2$$

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -5 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (\text{bij eigenwaarde } 2)$$

$$\begin{cases} x - y + z = 0 \\ -x + y - 5z = 0 \end{cases} \quad \begin{cases} y = x + z \\ z = 0 \end{cases} \quad \begin{cases} y = x \\ z = 0 \end{cases}$$

→ eigenvector  $\{ (a, a, 0) \mid a \in \mathbb{R}_0 \}$

$$\begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & -5 \\ 0 & 0 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (\text{bij eigenwaarde } 4)$$

$$\begin{cases} -x - y + z = 0 \\ -x - y - 5z = 0 \\ -6z = 0 \end{cases} \quad \begin{cases} -x = y \\ -x = y \\ z = 0 \end{cases}$$

→ eigenvector  $\{(a, -a, 0) \mid a \in \mathbb{R}_0\}$

$$\begin{pmatrix} 5 & -1 & 1 \\ -1 & 5 & -5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (\text{bij eigenwaarde } -2)$$

$$\begin{cases} 5x - y + z = 0 \\ -x + 5y - 5z = 0 \end{cases} \quad \begin{cases} y = 5x + z \\ 24x = 0 \end{cases} \quad \begin{cases} y = z \\ x = 0 \end{cases}$$

→ eigenvector  $\{(0, a, a) \mid a \in \mathbb{R}_0\}$

Een basis van  $\mathbb{R}^3$  die we vormen met de eigenvectoren kan dan zijn  $\{(0, 1, 1), (1, -1, 0), (1, 1, 0)\}$ .

Nu, door recursie vinden we gemakkelijk dat

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \left(\frac{1}{4}\right)^n \cdot \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & -5 \\ 0 & 0 & -2 \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \\ = \left(\frac{1}{4}\right)^n \cdot \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & -5 \\ 0 & 0 & -2 \end{pmatrix}^n \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{Nu is } \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

En daarom is

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \left(\frac{1}{4}\right)^n \left( A^n \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + A^n \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + A^n \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right)$$

En omdat  $A^n X = \lambda^n X$  met  $\lambda$  is een eigenwaarde van  $A$  en  $X$  de bijhorende eigenvector

geldt dat

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} &= \left(\frac{1}{4}\right)^n \left( \begin{pmatrix} 0 \\ (-2)^n \\ (-2)^n \end{pmatrix} + \begin{pmatrix} 4^n \\ -4^n \\ 0 \end{pmatrix} + \begin{pmatrix} 2^n \\ 2^n \\ 0 \end{pmatrix} \right) \\ &= \left(\frac{1}{4}\right)^n \begin{pmatrix} 4^n + 2^n \\ (-2)^n - 4^n + 2^n \\ (-2)^n \end{pmatrix} \\ &= \begin{pmatrix} (4^n + 2^n) / (4^n) \\ ((-2)^n - 4^n + 2^n) / (4^n) \\ (-2)^n / (4^n) \end{pmatrix} \end{aligned}$$

en daarom is:

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(\frac{4}{4}\right)^n + \left(\frac{2}{4}\right)^n = 1$$

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} \left(\frac{-2}{4}\right)^n - \left(\frac{4}{4}\right)^n + \left(\frac{2}{4}\right)^n = -1$$

$$\lim_{n \rightarrow \infty} z_n = \lim_{n \rightarrow \infty} \left(\frac{-2}{4}\right)^n = 0$$

11. a)

$$A = \begin{bmatrix} 0,6 & 3,5 & 2,4 & 1,2 \\ 0,5 & 0 & 0 & 0 \\ 0 & 0,75 & 0 & 0 \\ 0 & 0 & 0,75 & 0 \end{bmatrix}$$

$$\det \begin{pmatrix} 0,6 - \lambda & 3,5 & 2,4 & 1,2 \\ 0,5 & -\lambda & 0 & 0 \\ 0 & 0,75 & -\lambda & 0 \\ 0 & 0 & 0,75 & -\lambda \end{pmatrix}$$

$$= -\frac{3}{4} \det \begin{pmatrix} 0,6-\lambda & 3,5 & 1,2 \\ 0,5 & -\lambda & 0 \\ 0 & 0,75 & 0 \end{pmatrix} - \lambda \det \begin{pmatrix} 0,6-\lambda & 3,5 & 2,4 \\ 0,5 & -\lambda & 0 \\ 0 & 0,75 & -\lambda \end{pmatrix} = 0$$

$$\Leftrightarrow -\frac{3}{4} \cdot \frac{6}{5} \cdot \left( \frac{1 \cdot 3}{2 \cdot 4} \right) - \lambda \left( -0,75(-1,2) - \lambda \left( (0,6-\lambda)(-\lambda) - 1,75 \right) \right) = 0$$

$$\Leftrightarrow -\frac{27}{80} - 0,9\lambda - 0,6\lambda^3 + \lambda^4 - 1,75\lambda^2 = 0$$

$$\Leftrightarrow \lambda^4 - 0,6\lambda^3 - 1,75\lambda^2 - 0,9\lambda - \frac{27}{80} = 0$$

We zoeken een nulpunt via de Newton-Raphsonmethode  
 Beschouw de functie  $f: \mathbb{R} \rightarrow \mathbb{R}: x \mapsto x^4 - 0,6x^3 - 1,75x^2 - 0,9x - \frac{27}{80}$   
 De Newton-Raphson recursie vergelijking wordt ons:

$$x_{n+1} = x_n - \frac{x_n^4 - 0,6x_n^3 - 1,75x_n^2 - 0,9x_n - \frac{27}{80}}{4x_n^3 - 1,8x_n^2 - 3,5x_n - 0,9}$$

Beginnen we met een lukraakgekozen waarde  $x = 1$ :

$n$	$x_n$	$f(x_n)$
0	1	-2,588
1	-0,81	-0,229
2	-0,8154	-0,0073
3	-0,8153	$1,7 \cdot 10^{-4}$
4	-0,8153	

Nemen we een andere beginwaarde  $x = 3$ :

$n$	$x_n$	$f(x_n)$
0	3	46,013
1	2,07	13,315
2	1,8998	3,3394
3	1,8588	0,54902
4	1,8565	0,02762
5	1,8565	

We ontbinden de functie nu met horner:

$$\begin{array}{r|rrrrr} & 1 & -0,6 & -1,75 & -0,9 & -0,3375 \\ -0,8153 & & -0,8153 & 1,1539 & 0,486 & 0,3375 \\ \hline & 1 & -1,4153 & -0,5961 & -0,414 & 0 \end{array}$$

$$(x + 0,8153)(x^3 - 1,4153x^2 - 0,5961x - 0,414)$$

$$\begin{array}{r|rrrr} & 1 & -1,4153 & -0,5961 & -0,414 \\ 1,8565 & & 1,8565 & 0,8191 & 0,414 \\ \hline & 1 & 0,4412 & 0,2230 & 0 \end{array}$$

$$(x + 0,8153)(x - 1,8565)(x^2 + 0,4412x + 0,223) = 0$$

$$D = -0,6973$$

$$x = \frac{-0,4412 \pm \sqrt{0,6973}}{2} i$$

$$= -0,2206 + 0,4175i$$

$$v - 0,2206 - 0,4175i$$

De 4 eigenwaarden zijn dus:

$$\lambda_1 = -0,82, \lambda_2 = 1,86, \lambda_3 = -0,22 + 0,42i, \lambda_4 = -0,22 - 0,42i$$

b) Deze matrix is diagonaliseerbaar omdat het een  $(4 \times 4)$  matrix is met 4 verschillende eigenwaarden.

c)  $|\lambda_1| = 0,82$   $|\lambda_2| = 1,86$   $|\lambda_3| = 0,474$   $|\lambda_4| = 0,474$   
we berekenen de eigenvector bij  $\lambda_2$

$$\begin{pmatrix} -1,26 & 3,5 & 2,4 & 1,2 \\ 0,5 & -1,86 & 0 & 0 \\ 0 & 0,75 & -1,86 & 0 \\ 0 & 0 & 0,75 & -1,86 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -1,26x + 3,5y + 2,4z + 1,2\mu = 0 \\ 0,5x - 1,86y = 0 \\ 0,75y - 1,86z = 0 \\ 0,75z - 1,86\mu = 0 \end{cases} \begin{cases} -0,2195y + 1,2\mu = 0 \\ x = 3,72y \\ z = 0,4032y \\ 0,3024y - 1,86\mu = 0 \end{cases}$$

$$\begin{cases} Y = 6,15 \mu \\ X = 22,88 \mu \\ Z = 2,48 \mu \\ Y = 6,15 \mu \end{cases} \rightarrow \text{eigenvector } \left\{ (22,88a, 6,15a, 2,48a, a) \mid a \in \mathbb{R} \right\}$$

$$d) \begin{pmatrix} x_n \\ y_n \\ z_n \\ u_n \end{pmatrix} = A^n \cdot a x_1 + A^n \cdot b \cdot x_2 + A^n \cdot c \cdot x_3 + A^n \cdot d \cdot x_4$$

$$\begin{aligned} &= a \lambda_1^n x_1 + b \lambda_2^n x_2 + c \lambda_3^n x_3 + d \lambda_4^n x_4 \\ &= a \cdot (-0,82)^n x_1 + b \cdot (1,86)^n \begin{pmatrix} 22,88 \\ 6,15 \\ 2,48 \\ 1 \end{pmatrix} + c \cdot (-0,22 + 0,42i)^n x_3 \\ &\quad + d \cdot (-0,22 - 0,42i)^n x_4 \end{aligned}$$

Omdat 1,86 de grootste modulus heeft zal in de limiet de andere termen verwaarloosbaar worden.

$$\lim_{n \rightarrow \infty} \begin{pmatrix} x_n \\ y_n \\ z_n \\ u_n \end{pmatrix} = \lim_{n \rightarrow \infty} (1,86)^n \begin{pmatrix} 22,88b \\ 6,15b \\ 2,48b \\ b \end{pmatrix} = +\infty$$

EIR jaar groeit de populatie met 1,86 op lange termijn  
Het relatieve aandeel:

$$\begin{aligned} \text{nuljarigen} & (22,88b) / (22,88b + 6,15b + 2,48b + b) \\ &= 0,70 \end{aligned}$$

$$\begin{aligned} \text{eenjarigen} & (6,15b) / (22,88b + 6,15b + 2,48b + b) \\ &= 0,19 \end{aligned}$$

$$\begin{aligned} \text{tweejarigen} & (2,48b) / (22,88b + 6,15b + 2,48b + b) \\ &= 0,08 \end{aligned}$$

$$\begin{aligned} \text{driejarigen} & (1b) / (22,88b + 6,15b + 2,48b + b) \\ &= 0,03 \end{aligned}$$

wanneer men de beginpopulatie schrijft als lineaire combinatie, mag de coëfficiënt bij eigenwaarde 1,86 niet nul zijn. Anders sterft de populatie uit.

12. Het gesloten lenontiefmodel karakteriseert zich zo:

$$AX = X \quad \text{en} \quad \sum_{i=1}^m a_{ij} = 1 \quad \text{voor alle } j = 1, 2, \dots, m$$

$A$  is dus een stochastische matrix.

Uit  $AX = X$  volgt dat 1 een eigenwaarde is bij eigenvector  $X$ . Stel  $X = (x_1, x_2, \dots, x_m)$ . Dan geldt volgens propositie 6.2.2.3. (3) dat  $(|x_1|, |x_2|, |x_3|, \dots, |x_m|)$  ook een eigen vector is bij eigenwaarde 1.

Dus  $A|X| = |X|$ . Er bestaat dus een economisch zinvolle oplossing.

### 1.3. Opdrachten

1. Zij  $F: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, y) \rightarrow xy^3 - 2xy^2 + 3x^2y + 6x^3$   
en  $c = 6$ .

Beschouw nu de functie  $h: \mathbb{R} \rightarrow \mathbb{R}^2: x \rightarrow (x, f(x))$

Beschouw nu de functie  $\tilde{f}: \mathbb{I} \rightarrow \mathbb{R}$  door  $\tilde{f} = F \circ h$

dan is  $\tilde{f} = F(x, f(x)) = 6$

Daarom is  $\tilde{f}' = 0$

Anderzijds is  $\tilde{f}' = D_1 F(x, f(x)) + D_2 F(x, f(x)) \cdot f'(x)$

Dus  $f'(1) = -\frac{D_1 F(1, 0)}{D_2 F(1, 0)}$

$$= -\frac{18}{3} = -6$$

Nu is  $\tilde{f}'' = D_{11}^2 F(x, f(x)) + D_{21}^2 F(x, f(x)) \cdot f'(x) + D_2 F(x, f(x)) \cdot f''(x)$

$$+ D_{22}^2 F(x, f(x)) \cdot f'(x) + D_{22}^2 F(x, f(x)) \cdot f'(x)^2$$

$$0 = \tilde{f}''(1) = 36 + 6 \cdot (-6) + 3 f''(1) + 6 \cdot (-6) + (-4) \cdot (-6)^2$$

$$0 = 3 f''(1) - 180$$

$$f''(1) = 60$$

2. Beschouw  $F: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, y) \rightarrow 3x^2 f(x, y) - x^2 y^2 + 2f(x, y)^3 - 3y f(x, y) - 1$

Wanneer we  $F(1, 1) = 0$  vinden we dat:

$$3f(1, 1) - 1 + 2f(1, 1)^3 - 3f(1, 1) - 1 = 0$$

$$f(1, 1)^3 = 1$$

$$f(1, 1) = 1$$

Nu is  $F(x, y) = 0$  voor alle  $(x, y) \in \mathbb{R}^2$

En daarom is  $D_1 F(x, y) = D_2 F(x, y) = D_{12}^2 F(x, y) = D_{22}^2 F(x, y) = D_{11}^2 F(x, y) = 0$

Nu is  $D_1 F(x, y) = 6x f(x, y) + 3x^2 D_1 f(x, y) - 2xy^2 + 6f(x, y)^2 D_1 f(x, y) - 3y D_1 f(x, y)$

$$\text{en } D_1 F(1, 1) = 6 + 3 D_1 f(1, 1) - 2 + 6 D_1 f(1, 1) - 3 D_1 f(1, 1) = 0$$

$$\Leftrightarrow D_1 f(1, 1) = -2/3$$

Nu is  $D_2 F(x, y) = 3x^2 D_2 f(x, y) - 2x^2 y + 6 D_2 f(x, y) f(x, y)^2 - 3f(x, y) - 3y D_2 f(x, y)$

$$\text{en } D_2 F(1, 1) = 3 D_2 f(1, 1) - 2 + 6 D_2 f(1, 1) - 3 - 3 D_2 f(1, 1) = 0$$

$$\Leftrightarrow D_2 f(1, 1) = 5/6$$

$$D_{2,1}^2 F(x,y) = 6x D_2 f(x,y) + 3x^2 D_{2,1}^2 f(x,y) - 4xy + 12 D_1 f(x,y) D_2 f(x,y) f(x,y) - 3 D_1 f(x,y) - 3y D_{2,1}^2 f(x,y) + 6 f(x,y)^2 D_{2,1}^2 f(x,y) = 0$$

$$D_{2,1}^2 F(1,1) = 5 + 3 D_{2,1}^2 f(1,1) - 4 - \frac{20}{3} + 2 - 3 D_{2,1}^2 f(1,1) + 6 D_{2,1}^2 f(1,1) = 0$$

$$\Leftrightarrow D_{2,1}^2 f(1,1) = 11/18$$

$$D_{2,2}^2 F(x,y) = 3x^2 D_{2,2}^2 f(x,y) - 2x^2 + 6 D_{2,2}^2 f(x,y) \cdot f(x,y)^2 + 12 D_2 f(x,y)^2 \cdot f(x,y) - 3 D_2 f(x,y) - 3 D_2 f(x,y) - 3y D_{2,2}^2 f(x,y) = 0$$

$$D_{2,2}^2 F(1,1) = 3 D_{2,2}^2 f(1,1) - 2 + 6 D_{2,2}^2 f(1,1) + 12 \cdot \frac{25}{36} - 5 - 3 D_{2,2}^2 f(1,1) = 0$$

$$\Leftrightarrow D_{2,2}^2 f(1,1) = -2/9$$

$$D_{1,1}^2 F(1,1) = 6 f(x,y) + 6x D_1 f(x,y) + 6x D_1 f(x,y) + 3x^2 D_{1,1}^2 f(x,y) - 2y^2 + 12 f(x,y) \cdot D_1 f(x,y)^2 + 6 f(x,y)^2 D_{1,1}^2 f(x,y) - 3y D_{1,1}^2 f(x,y)$$

$$D_{1,1}^2 F(1,1) = 6 - 4 - 4 + 3 D_{1,1}^2 f(1,1) - 2 + \frac{16}{3} + 6 D_{1,1}^2 f(1,1) - 3 D_{1,1}^2 f(1,1)$$

$$\Leftrightarrow D_{1,1}^2 f(1,1) = -2/9$$

3. Omdat  $\left(\frac{1}{2}k^{1/2} + \frac{1}{2}L^{1/2}\right)^2 = c$   
 is de afgeleide naar  $k$  gelijk aan nul. Bovendien is  
 deze ook gelijk aan

$$\left(\frac{1}{4}k^{-1/2}\right) = 0$$

Beschouw de functie  $h: \mathbb{R} \rightarrow \mathbb{R}^2: k \rightarrow (k, L(k))$   
 en de functie  $\tilde{L}: \mathbb{R} \rightarrow \mathbb{R}$  door  $Q \circ k = Q(k, L(k)) = c$   
 Dus  $\tilde{L}' = 0$

$$\text{Bovendien is } \tilde{L}' = D_1 Q(k, L(k)) + D_2 Q(k, L(k)) \cdot L'(k)$$

$$= 2 \left(\frac{1}{2}k^{1/2} + \frac{1}{2}L^{1/2}\right) \left(\frac{1}{4}k^{-1/2}\right) +$$

$$2 \left(\frac{1}{2}k^{1/2} + \frac{1}{2}L^{1/2}\right) \left(\frac{1}{4}L^{-1/2}\right) \cdot L'(k) = 0$$

$$\Leftrightarrow L'(k) = \frac{-\sqrt{L}}{\sqrt{k}} = -\frac{D_1 Q(k, L)}{D_2 Q(k, L)}$$

4. Beschouw  $F: \mathbb{R}^{n+1} \rightarrow \mathbb{R} : (x_1, x_2, \dots, x_n, y) \rightarrow F(x_1, x_2, \dots, x_n, y)$   
 en  $f: \mathbb{R}^n \rightarrow \mathbb{R} : (x_1, x_2, \dots, x_n) \rightarrow f(x_1, x_2, \dots, x_n)$

Beschouw nu de functie  $h: \mathbb{R}^n \rightarrow \mathbb{R}^{n+1} : x \rightarrow (x, f(x))$

en de functie  $\tilde{f}: \mathbb{R}^n \rightarrow \mathbb{R}$ : door  $\tilde{f} = F \circ h = F(x_1, x_2, \dots, x_n, f(x_1, x_2, \dots, x_n)) = c$

Omdat  $\tilde{f}$  een constante functie is, is  $\tilde{f} \stackrel{\sim}{=} D_i f(x) = 0$  voor alle  $i=1, \dots, n$ .

Bovendien geldt volgens de kettingregel:

$$D_i \tilde{f}(x_1, x_2, \dots, x_n) = D_i F(x_1, \dots, x_n, f(x_1, \dots, x_n)) + D_{n+1} F(x_1, \dots, x_n, f(x_1, \dots, x_n)) \cdot D_i f(x) = 0$$

$$\text{Hieruit volgt: } D_i f(x) = - \frac{\partial F}{\partial x_i}(x_1, x_2, x_3, \dots, x_n, f(x_1, x_2, \dots, x_n))$$

$$\frac{\partial F}{\partial y}(x_1, x_2, x_3, \dots, x_n, f(x_1, x_2, \dots, x_n))$$

5. Herschrijven we 1.2.4 met  $m=1$  bekomen we:

Zij  $F: \mathbb{R}^{n+1} \rightarrow \mathbb{R} : (x_1, x_2, \dots, y) \rightarrow F(x, y)$  een functie die continue partiële afgeleiden heeft. Zij  $c \in \mathbb{R}$  en  $(x^*, y^*) \in \mathbb{R}^{n+1}$  een punt waarvoor  $F(x^*, y^*) = c$ . Nu is  $dy F(x^*, y^*) = \frac{\partial F}{\partial y}(x^*, y^*)$ .

Dit moet inverteerbaar zijn, of nog het me  $\frac{\partial F}{\partial y}$  niet gelijk zijn aan nul. Dan bestaat er een open  $A \subseteq \mathbb{R}^n$  met  $x^* \in A$  en een open  $B \subseteq \mathbb{R}$  met  $y^* \in B$  zo dat  $\frac{\partial F}{\partial y}(x^*, y^*) \neq 0$  voor alle  $x \in A$  en  $y \in B$  en zo dat er een unieke functie

$f: A \subseteq \mathbb{R}^n \rightarrow B \subseteq \mathbb{R} : x \rightarrow f(x)$  bestaat die voldoet aan

$$(1) f(x^*) = y^*$$

$$(2) \forall x \in A : F(x, f(x)) = c$$

Bovendien heeft  $f$  continue partiële afgeleiden; voor alle

$$x \in A \text{ is } \frac{\partial f}{\partial x_i}(x) = - \frac{\frac{\partial F}{\partial x_i}(x, y)}{\frac{\partial F}{\partial y}(x, y)}$$

$$6. a) \begin{cases} u-v=6 \\ u=v^3 \end{cases}$$

$$\begin{cases} v^3-v-6=0 \\ u=v^3 \end{cases}$$

$$\begin{cases} v=2 \\ u=8 \end{cases}$$

b) Beschouw de functie  $F: \mathbb{R}^4 \rightarrow \mathbb{R}^2 : (x, y, u, v) \rightarrow \begin{cases} u-v-3x \\ u-v^3-x+2y \end{cases}$

en een  $c = (0, 0)$ . Dan kunnen we de algemere IFS toepassen als aan drie voorwaarden voldaan is:

①  $F$  heeft continue partiële afgeleiden: check.

②  $\exists (x, y, u, v) : F(x, y, u, v) = c$

check:  $F(2, 1, 8, 2) = (0, 0)$

③  $dy F(2, 1, 8, 2)$  is inverteerbaar

check:  $\begin{pmatrix} -1 & -1 \\ 1 & -12 \end{pmatrix}$  is inverteerbaar want  $\det = -12 + 1 = -11$

Dan geldt volgens de IFS dat er een open  $A \subseteq \mathbb{R}^2$

bestaat met  $(2, 1) \in A$  en een open  $B$  bestaat met

$(8, 2) \in B$  zo dat er een functie  $f: A \rightarrow B : (x, y) \rightarrow f(x, y)$

die voldoet aan  $\hat{u}$  unieke

(1)  $f(2, 1) = (8, 2)$

(2)  $\forall x \in A : F(x, y, f(x, y)) = c$

c) Beschouw de functie  $h: \mathbb{R}^2 \rightarrow \mathbb{R}^4 : (x, y) \rightarrow (x, y, F_1(x, y), F_2(x, y))$

en de functie  $\tilde{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2 : \text{door } \tilde{F} = F \circ h = F(x, y, F_1(x, y), F_2(x, y)) = ($

Omdat  $D_1 \tilde{F}_i = D_2 \tilde{F}_i = D_{12}^2 \tilde{F}_i = D_{22}^2 \tilde{F}_i = D_{11}^2 \tilde{F}_i = 0$  voor alle  $i = 1, 2$  geldt:

$$D_1 \tilde{F}_1 = D_1 F_1(x, y, F_1(x, y), F_2(x, y)) + D_3 F_1(x, y, F_1(x, y), F_2(x, y)) \cdot D_1 F_1(x, y) + D_4 F_1(x, y, F_1(x, y), F_2(x, y)) \cdot D_1 F_2(x, y) = 0$$

$$D_1 \tilde{F}_2 = D_1 F_2(x, y, F_1(x, y), F_2(x, y)) + D_3 F_2(x, y, F_1(x, y), F_2(x, y)) \cdot D_1 F_1(x, y) + D_4 F_2(x, y, F_1(x, y), F_2(x, y)) \cdot D_1 F_2(x, y) = 0$$

Hieruit volgt het stelsel wanneer we voor  $(x, y) = (2, 1)$  invullen:

$$\begin{cases} -3 + D_1 F_1(2, 1) - D_1 F_2(2, 1) = 0 & \begin{cases} D_1 F_1(2, 1) = 35/11 \\ D_2 F_2(2, 1) = 2/11 \end{cases} \\ -1 + D_1 F_1(2, 1) - 3 \cdot 4 \cdot D_1 F_2(2, 1) = 0 \end{cases}$$

$$D_2 \tilde{F}_1 = D_2 F_1(x, y, F(x, y)) + D_3 F_1(x, y, F(x, y)) \cdot D_2 F_1(x, y) + D_4 F_1(x, y, F(x, y)) \cdot D_2 F_2(x, y)$$

$$D_2 \tilde{F}_2 = D_2 F_2(x, y, F(x, y)) + D_3 F_2(x, y, F(x, y)) \cdot D_2 F_1(x, y) + D_4 F_2(x, y, F(x, y)) \cdot D_2 F_2(x, y)$$

Wanneer we voor  $(x, y) = (2, 1)$  invullen, krijgen we het stelsel:

$$\begin{cases} D_2 F_1(2, 1) - D_2 F_2(2, 1) = 0 \\ 2 + D_2 F_1(2, 1) - 12 D_2 F_2(2, 1) = 0 \end{cases} \quad \begin{cases} D_2 F_1(2, 1) = 2/11 \\ D_2 F_2(2, 1) = 2/11 \end{cases}$$

## 2.2.5. Operatoren

1. eigenwaardemethode:

$$\det \begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 2-\lambda & 3 \\ 1 & 3 & 6-\lambda \end{pmatrix}$$

$$= (1-\lambda)[(2-\lambda)(6-\lambda)-9] - [(6-\lambda) \cdot 3] + [3 - (2-\lambda)]$$

$$= (1-\lambda)[\lambda^2 - 8\lambda + 3] + \lambda - 3 + 3 - 2 + \lambda$$

$$= \lambda^2 - 8\lambda + 3 - \lambda^3 + 8\lambda^2 - 3\lambda + 2\lambda - 2$$

$$= -\lambda^3 + 9\lambda^2 - 9\lambda + 1$$

$$\begin{array}{c|cccc} & -1 & 9 & -9 & 1 \\ 1 & & -1 & 8 & -1 \\ \hline & -1 & 8 & -1 & 0 \end{array}$$

$$(\lambda - 1)(-\lambda^2 + 8\lambda - 1)$$

$$D = 64 - 4 = 60$$

$$\lambda = 1 \vee \lambda = \frac{-8 + \sqrt{60}}{-2} \vee \lambda = \frac{-8 - \sqrt{60}}{-2}$$

$$\lambda = 1 \vee \lambda = 0,127 \vee \lambda = 7,873$$

→ alle eigenwaarden  $> 0$  → positief definit

determinantmethode

$$\det(1) = 1$$

$$\det \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = 2 - 1 = 1$$

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix} = 12 - 9 - 6 + 3 + 3 - 2 = 1$$

→ positief definit

2.  $\begin{pmatrix} 1 & 7 & 3 \\ 7 & 4 & 5 \\ 3 & 5 & 2 \end{pmatrix}$  det *in rijmethode*

$$\det(1) = 1$$

$$\det \begin{pmatrix} 1 & 7 \\ 7 & 4 \end{pmatrix} = 4 - 49 = -45$$

→ nog positief definit, nog negatief definit

3. in matrixvorm

$$\begin{pmatrix} 1 & -a & 1 \\ -a & a^2+b & -a-3b \\ 1 & -a-3b & 1+9b+c \end{pmatrix}$$

$\det(1) = 1$  → *nog negatief definit*

$$\det \begin{pmatrix} 1 & -a \\ -a & a^2+b \end{pmatrix} = a^2+b - a^2 = b$$

als  $b \leq 0$  → *nog positief definit, nog negatief.*

$b > 0$

$$\det \begin{pmatrix} 1 & -a & 1 \\ -a & a^2+b & -a-3b \\ 1 & -a-3b & 1+9b+c \end{pmatrix}$$

$$= (a^2+b)(1+9b+c) - (a^2+6ab+9b^2)$$

$$+ a[(-a)(1+9b+c) + (a+3b)]$$

$$+ (a)(a+3b) - (a^2+b)$$

$$= a^2 + 9a^2b + a^2c + b + 9b^2 + bc - a^2 - 6ab - 9b^2$$

$$= a^2 - 9a^2b - a^2c + a^2 + 3ab + a^2 + 3ab - a^2 - b$$

$$= bc \quad \rightarrow \text{als } c > 0 \text{ positief definit}$$

anders *nog positief nog negatief*

4. a) NEEN bv.

$$\begin{pmatrix} 10 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\det(10) > 0$$

$$\det \begin{pmatrix} 10 & -1 \\ -1 & 1 \end{pmatrix} = 10 + 1 > 0$$

$$\det \begin{pmatrix} 10 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= 10(2-1) + 1(-2+1) + 1(-1-1)$$

$$= 10 - 1 - 2 = 7$$

→ positief definit

b) NEEN bv

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\det(1) > 0$$

$$\det \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = 1 - 4 = -3$$

→ nog positief, nog negatief definit.

## 2.3.5. Opdrachten

1 a)  $D_3 f(x, y, z) = 3z + x + y$

$D_2 f(x, y, z) = 3y - x + z$

$D_1 f(x, y, z) = 3x - y + z$

$$\begin{cases} 3z + x + y = 0 \\ 3y - x + z = 0 \\ 3x - y + z = 0 \end{cases} \quad \begin{cases} y = -3z - x \\ -8z - 4x = 0 \\ 4x + 4z = 0 \end{cases}$$

$$\begin{cases} y = -3z - x \\ -8z + 4z = 0 \\ x = -z \end{cases} \quad \begin{cases} y = 0 \\ z = 0 \\ x = 0 \end{cases}$$

$H_{(0,0,0)} = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \rightarrow \text{convex}$

$\det H_{(0,0,0)} = 3 \cdot (9 - 1) + (-3 \cdot -1) + (-1 \cdot -3)$   
 $= 24 - 4 - 4 = 16$

$\det \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} = 9 - 1 = 8$

$\det(3) = 3$

$\rightarrow$  positief definit  $\rightarrow$  minimum  $\rightarrow$  globaal minimum.

b)  $D_1 f(x, y, z) = 2x - 2yz$

$D_2 f(x, y, z) = 2y - 2xz$

$D_3 f(x, y, z) = 2z - 2xy$

$$\begin{cases} x = yz \\ y = yz^2 \\ z = y^2z^3 \end{cases} \quad \text{stel } y \neq 0 \quad \begin{cases} x = 1 \vee x = -1 \\ z = 1 \vee \\ y = 1 \vee y = -1 \end{cases} \quad \begin{cases} x = -1 \vee x = 1 \\ z = -1 \\ y = -1 \vee y = -1 \end{cases}$$

stel  $y = 0$

$$\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$

critieke punten  $(0,0,0) \wedge (1,1,1) \wedge (1,-1,-1) \wedge (-1,-1,1) \wedge (-1,1,-1)$

$$H(x,y,z) = \begin{pmatrix} 2 & -2z & -2y \\ -2z & 2 & -2x \\ -2y & -2x & 2 \end{pmatrix}$$

$$H_{(0,0,0)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\det(H_{(0,0,0)} - \lambda \mathbb{1}) = (2-\lambda)^3$$

→ eigenwaarde = 2

→ positief definit → minimum

$$H_{(1,1,1)} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$

$$\det = 2(4-4) + 2(-4-4) - 2(4+4)$$

$$= -32$$

$$\det H_2 = 4-4 = 0$$

Dan volgt uit gevolg 2.2.4.3 dat er minstens  
1 negatieve en 1 positieve eigenwaarde bestaat  
→ geen extremum

$$H_{(1,-1,-1)} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix}$$

$$\det = 2(4-4) - 2(4+4) + 2(-4-4)$$

$$= -32$$

$$\det H_2 = 4-4 = 0$$

→ geen extremum

$$H_{(-1,-1,1)} = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

$$\det = 2(4-4) + 2(-4-4) + 2(-4-4)$$

$$= -32$$

$$\det H_2 = 4-4 = 0$$

→ geen extremum

$$H_{(-1,1,-1)} = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 2 & 2 \\ -2 & 2 & 2 \end{pmatrix}$$

$$\det = 2(4-4) - 2(4+4) - 2(4+4) = -32$$

$$\det H_2 = 4-4 = 0$$

→ geen extremum

$$\begin{aligned} c) D_1 f(x,y,z) &= 2xe^{-xy} - yx^2e^{-xy} - y(y^2+z^2)e^{-xy} \\ &= e^{-xy}(2x - yx^2 - y^3 - yz^2) \\ D_2 f(x,y,z) &= e^{-xy}(2y - x^3 - xy^2 - xz^2) \\ D_3 f(x,y,z) &= 2ze^{-xy} \end{aligned}$$

$$\begin{cases} e^{-xy}(2x - yx^2 - y^3 - yz^2) = 0 \\ e^{-xy}(2y - x^3 - xy^2 - xz^2) = 0 \\ 2ze^{-xy} = 0 \end{cases} \quad \begin{cases} 2x - yx^2 - y^3 = 0 \\ 2y - x^3 - xy^2 = 0 \\ z = 0 \end{cases}$$

$$\begin{cases} 2x - y(x^2 + y^2) = 0 \\ 2y - x(x^2 + y^2) = 0 \\ z = 0 \end{cases} \quad \textcircled{1} y \neq 0 \quad \begin{cases} x^2 + y^2 = 2x/y \\ 2y - 2x^2/y = 0 \\ z = 0 \end{cases}$$

$$\begin{cases} 2y^3 = 2x \\ y^2 = x^2 \\ z = 0 \end{cases} \quad \begin{cases} x = y^3 \\ y^2 = y^6 \\ z = 0 \end{cases} \quad \begin{cases} x = 1 \\ y = 1 \\ z = 0 \end{cases} \quad \vee \quad \begin{cases} x = -1 \\ y = -1 \\ z = 0 \end{cases}$$

$$\textcircled{2} y = 0 \quad \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$

→ Drie kritieke punten  $(0,0,0)$ ,  $(1,1,0)$  en  $(-1,-1,0)$

$$D_{11}^2 f(x,y,z) = 2e^{-xy} - 2xye^{-xy} - 2yx e^{-xy} + y^2x^2e^{-xy} + y^4e^{-xy} + y^2z^2e^{-xy}$$

$$D_{22}^2 f(x,y,z) = e^{-xy}(-2xy + x^4 + x^2y^2 + x^2z^2) + e^{-xy}(2 \cdot 2xy)$$

$$D_{33}^2 f(x,y,z) = 2e^{-xy}$$

$$D_{21}^2 f(x,y,z) = e^{-xy}(-2x^2 + yx^3 + xy^3 + xyz^2) + e^{-xy}(-x^2 - 3y^2 - z^2)$$

$$D_{31}^2 f(x,y,z) = -2yz e^{-xy}$$

$$D_{32}^2 f(x,y,z) = -2xz e^{-xy}$$

$$H_{(0,0,0)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

→ positief definit

→ minimum

$$H_{(1,1,0)} = \frac{1}{e} \begin{pmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\det = 4(-8) = -32$$

$$\det H_2 = -16$$

→ geen extremum

$$H_{(-1,-1,0)} = \frac{1}{e} \begin{pmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\det = 4(-8) = -32$$

$$\det H_2 = -16$$

→ geen extremum

$$2. D_1 g(x, y, z) = f'(x^2 + 2y^2 + 3z^2) \cdot 2x$$

$$D_2 g(x, y, z) = f'(x^2 + 2y^2 + 3z^2) \cdot 4y$$

$$D_3 g(x, y, z) = f'(x^2 + 2y^2 + 3z^2) \cdot 6z$$

$$\text{voor } (0, 0, 0) \text{ is } D_1 g = D_2 g = D_3 g = 0$$

$$D_{11}^2 g(x, y, z) = 2 f''(x^2 + 2y^2 + 3z^2) + 2x \cdot f'''(x^2 + 2y^2 + 3z^2)$$

$$D_{22}^2 g(x, y, z) = 4 f''(x^2 + 2y^2 + 3z^2) + 4y \cdot f'''(x^2 + 2y^2 + 3z^2)$$

$$D_{33}^2 g(x, y, z) = 6 f''(x^2 + 2y^2 + 3z^2) + 6z \cdot f'''(x^2 + 2y^2 + 3z^2)$$

$$D_{21}^2 g(x, y, z) = 2x \cdot f'''(x^2 + 2y^2 + 3z^2) \cdot 4y$$

$$D_{31}^2 g(x, y, z) = 2x \cdot f'''(x^2 + 2y^2 + 3z^2) \cdot 6z$$

$$D_{32}^2 g(x, y, z) = 4y \cdot f'''(x^2 + 2y^2 + 3z^2) \cdot 6z$$

$$H_{(0,0,0)} = \begin{pmatrix} 2f'(0) & 0 & 0 \\ 0 & 4f'(0) & 0 \\ 0 & 0 & 6f'(0) \end{pmatrix}$$

→ alle EW < 0

→ lokaal maximum

3.

## 2.45 opdrachten

1.  $\left\{ \begin{array}{l} \text{maximaliseer } x^2 + y^2 \\ \text{onder de RV } (x-3)^2 + (y-4)^2 = 1 \end{array} \right.$

De punten waarvoor  $\nabla g = (0,0)$  zijn

$$(2x-6, 2y-8) = (0,0)$$

$$x=3 \text{ en } y=4$$

maar voldoet niet aan RV

Lagrangefunctie

$$L: \mathbb{R}^3 \rightarrow \mathbb{R}: (x, y, \lambda) \rightarrow x^2 + y^2 - \lambda((x-3)^2 + (y-4)^2 - 1)$$

$$\left\{ \begin{array}{l} D_1 L(x, y, \lambda) = 2x - 2\lambda x + 6\lambda = 0 \\ D_2 L(x, y, \lambda) = 2y - 2\lambda y + 8\lambda = 0 \\ D_3 L(x, y, \lambda) = -(x-3)^2 - (y-4)^2 + 1 = 0 \end{array} \right.$$

Merk op dat  $x \neq 0$  en  $y \neq 0$  want dan is niet aan de RV voldaan.

$$\left\{ \begin{array}{l} \lambda = \frac{2x}{2x-6} \\ \lambda = \frac{2y}{2y-8} \\ -(x-3)^2 - (y-4)^2 + 1 = 0 \end{array} \right. \quad \left\{ \begin{array}{l} 2x(2y-8) = 2y(2x-6) \\ \dots \\ \dots \end{array} \right.$$

$$\left\{ \begin{array}{l} x = \frac{3}{4}y \\ \lambda = (2y)/(2y-8) \\ -\frac{9}{16}y^2 - 9 + \frac{9}{2}y - y^2 - 16 + 8y + 1 = 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} x = 3y/4 \\ \lambda = (2y)/(2y-8) \\ \frac{-25}{16}y^2 + \frac{25y}{2} - 24 = 0 \\ D = \frac{625}{4} - 150 \end{array} \right.$$

$$y = \frac{-6,25 \pm 2,5}{-3,125} = \left\{ \begin{array}{l} 4,8 \\ 3,2 \end{array} \right.$$

$$\left\{ \begin{array}{l} y = 4,8 \\ x = 3,6 \\ \lambda = 6 \end{array} \right. \quad \vee \quad \left\{ \begin{array}{l} y = 3,2 \\ x = 2,4 \\ \lambda = -4 \end{array} \right.$$

we stellen de Hessiaan op van  $L$

$$H_a = \begin{pmatrix} 2-2\lambda & 0 & -2x+6 \\ 0 & 2-2\lambda & -2y+8 \\ -2x+6 & -2y+8 & 0 \end{pmatrix}$$

$$H_{(3,6; 4,8; 6)} = \begin{pmatrix} -10 & 0 & -1,2 \\ 0 & -10 & -1,6 \\ -1,2 & -1,6 & 0 \end{pmatrix}$$

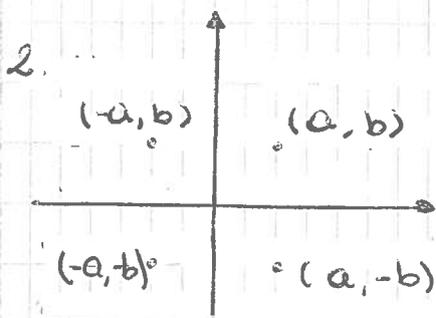
$$\det H_1 = -10 \left( -(1,6)^2 \right) - 1,2 \left( -(-1,2) \cdot (-10) \right) \\ = 40$$

→ maximum  $(-1)^1 = -1$

$$H_{(2,4; 3,2; -4)} = \begin{pmatrix} 10 & 0 & 1,2 \\ 0 & 10 & 0,8 \\ 1,2 & 0,8 & 0 \end{pmatrix}$$

$$\det H_1 = 10 \left( -0,8^2 \right) + 1,2 \left( -1,2 \cdot 10 \right) \\ = -20,8$$

→ minimum  $(-1)^m = -1$



De afstand tussen  $(-a, b)$  en  $(a, b)$  is  $\sqrt{4a^2} = 2a$

De afstand tussen  $(a, b)$  en  $(a, -b)$  is  $\sqrt{4b^2} = 2b$

De oppervlakte is dus  $2a \cdot 2b = 4ab$

Behalve dat moeten deze punten allemaal op een cirkel liggen. De afstand tot het middelpunt (oorsprong) moet dus gelijk zijn aan  $r$ :  $\sqrt{a^2+b^2} = r$

$$\Leftrightarrow a^2 + b^2 = r^2$$

Het probleem wordt dus maximaliseer  $4ab$

onder de RV  $a^2 + b^2 = r^2$

De Lagrangefunctie is:

$$L: \mathbb{R}^3 \rightarrow \mathbb{R}: (a, b, \lambda) \rightarrow 4ab - \lambda(a^2 + b^2 - r^2)$$

$$\begin{cases} D_1 L(a, b, \lambda) = 0 = 4b - 2a\lambda \\ D_2 L(a, b, \lambda) = 0 = 4a - 2b\lambda \\ D_3 L(a, b, \lambda) = 0 = -a^2 - b^2 + r^2 \end{cases}$$

Omdat  $a \neq 0$  en  $b \neq 0$  geldt:

$$\begin{cases} \lambda = \frac{a}{2b} \\ \lambda = \frac{b}{2a} \\ a^2 + b^2 - r^2 = 0 \end{cases} \quad \begin{cases} b = a \\ \lambda = 1/2 \\ a = \sqrt{\frac{r^2}{2}} \end{cases}$$

Dan is de oppervlakte =  $4 \cdot \sqrt{\frac{r^2}{2}} \cdot \sqrt{\frac{r^2}{2}} = 4 \cdot \frac{r^2}{2} = 2r^2$   
is dit een maximum?

$$H(a, b, \lambda) = \begin{pmatrix} -2\lambda & 4 & -2a \\ 4 & -2\lambda & -2b \\ -2a & -2b & 0 \end{pmatrix}$$

$$H\left(\sqrt{\frac{r^2}{2}}, \sqrt{\frac{r^2}{2}}, \frac{1}{2}\right) = \begin{pmatrix} -1 & 4 & -2\sqrt{\frac{r^2}{2}} \\ 4 & -1 & -2\sqrt{\frac{r^2}{2}} \\ -2\sqrt{\frac{r^2}{2}} & -2\sqrt{\frac{r^2}{2}} & 0 \end{pmatrix}$$

$$\det H = -1 \left(-4 \frac{r^2}{2}\right) - 4 \left(-4 \frac{r^2}{2}\right) - 2\sqrt{\frac{r^2}{2}} \left(-8 \sqrt{\frac{r^2}{2}} - 2\sqrt{\frac{r^2}{2}}\right)$$

$$= 2r^2 + 8r^2 + 10r^2 = 20r^2 > 0$$

is gelijk aan het teken van  $(-1)^{m+1}$

→ maximum

3. maximaliseer  $(x-1)^2 + (y-2)^2 + (z-3)^2$

onder de RV:  $x^2 + y^2 + z^2 = 1$

De punten waarvoor  $\nabla g = (0, 0, 0)$  is

$$(2x, 2y, 2z) = (0, 0, 0)$$

$$(x, y, z) = (0, 0, 0)$$

→ voldoet niet aan RV

De Lagrangefunctie is:

$$L: \mathbb{R}^4 \rightarrow \mathbb{R}: (x, y, z, \lambda) \rightarrow -2x - 4y - 6z + 15 - \lambda(x^2 + y^2 + z^2 - 1)$$

$$\begin{cases} D_1 L(x, y, z, \lambda) = 0 = -2 - 2x\lambda \\ D_2 L(x, y, z, \lambda) = 0 = -4 - 2y\lambda \\ D_3 L(x, y, z, \lambda) = 0 = -6 - 2z\lambda \\ D_4 L(x, y, z, \lambda) = 0 = -x^2 - y^2 - z^2 + 1 \end{cases}$$

$$\begin{cases} \lambda = -1/x \\ \lambda = -2/y \\ \lambda = -3/z \\ x^2 + y^2 + z^2 = 1 \end{cases} \quad \begin{cases} y = 2x \\ 3y = 2z \\ z = 3x \\ x^2 + 4x^2 + 9x^2 = 1 \end{cases}$$

$$\begin{cases} \lambda = \sqrt{14} \\ z = -3(1/14)^{1/2} \\ y = -2(1/14)^{1/2} \\ x = -(1/14)^{1/2} \end{cases} \quad \begin{cases} \lambda = -\sqrt{14} \\ z = 3(1/14)^{1/2} \\ y = 2(1/14)^{1/2} \\ x = (1/14)^{1/2} \end{cases}$$

Hessiaan:

$$\begin{pmatrix} -2\lambda & 0 & 0 & -2x \\ 0 & -2\lambda & 0 & -2y \\ 0 & 0 & -2\lambda & -2z \\ -2x & -2y & -2z & 0 \end{pmatrix}$$

$$H(-\sqrt{\frac{1}{14}}, -2\sqrt{\frac{1}{14}}, -3\sqrt{\frac{1}{14}}, \sqrt{14}) = \begin{pmatrix} -2\sqrt{14} & 0 & 0 & 2\sqrt{\frac{1}{14}} \\ 0 & -2\sqrt{14} & 0 & 4\sqrt{\frac{1}{14}} \\ 0 & 0 & -2\sqrt{14} & 6\sqrt{\frac{1}{14}} \\ 2\sqrt{\frac{1}{14}} & 4\sqrt{\frac{1}{14}} & 6\sqrt{\frac{1}{14}} & 0 \end{pmatrix}$$

$$\begin{aligned} \Delta_2 &= -2\sqrt{14} \left(-\frac{36}{14}\right) + 4\sqrt{\frac{1}{14}}(8) \\ &= \frac{36}{7}\sqrt{14} + 32\sqrt{\frac{1}{14}} > 0 \end{aligned}$$

$$\begin{aligned} \Delta_1 &= -2\sqrt{14} \left(\frac{36}{7}\sqrt{14} + 32\sqrt{\frac{1}{14}}\right) - 2\sqrt{\frac{1}{14}}(2\sqrt{14} \cdot 4) \\ &= -144 - 64 - 16 = -224 < 0 \end{aligned}$$

Afwisselend teken te beginnen met  $(-1)^{m+1}$

→ maximum

$$H\left(\sqrt{\frac{1}{14}}, 2\sqrt{\frac{1}{14}}, 3\sqrt{\frac{1}{14}}, -\sqrt{14}\right) = \begin{pmatrix} 2\sqrt{14} & 0 & 0 & -2\sqrt{\frac{1}{14}} \\ 0 & 2\sqrt{14} & 0 & -4\sqrt{\frac{1}{14}} \\ 0 & 0 & 2\sqrt{14} & -6\sqrt{\frac{1}{14}} \\ -2\sqrt{\frac{1}{14}} & -4\sqrt{\frac{1}{14}} & -6\sqrt{\frac{1}{14}} & 0 \end{pmatrix}$$

$$\Delta_2 = 2\sqrt{14} \left(-\frac{36}{14}\right) - 4\sqrt{\frac{1}{14}} (8)$$

$$= \frac{-36}{7}\sqrt{14} - 32\sqrt{\frac{1}{14}} < 0$$

$$\Delta_1 = 2\sqrt{14} \left(\frac{-36}{7}\sqrt{14} - 32\sqrt{\frac{1}{14}}\right) + 2\sqrt{\frac{1}{14}} (-2\sqrt{14} \cdot 4)$$

$$= -144 - 64 - 16$$

$$= -224 < 0$$

zelfde teken gelijk aan  $(-1)^m$   
 $\rightarrow$  minimum.

4. minimaliseer  $x^2 + y^2 + z^2$   
 onder de RV:  $\begin{cases} x^2 - xy + y^2 - z^2 = 1 \\ x^2 + y^2 = 1 \end{cases}$

Door substitutie via tweede RV bekomen we

minimaliseer  $1 + z^2$   
 onder de RV:  $\begin{cases} z^2 + xy = 0 \\ x^2 + y^2 = 1 \end{cases}$

$$\nabla g_1 = (0, 0, 0) \Leftrightarrow (y, x, 2z) = (0, 0, 0)$$

$\rightarrow$  niet in RV

$$\nabla g_2 = (0, 0, 0) \Leftrightarrow (2x, 2y, 0) = (0, 0, 0)$$

$\rightarrow$  niet in RV

de Lagrangefunctie is

$$L: \mathbb{R}^5 \rightarrow \mathbb{R}: (x, y, z, \lambda_1, \lambda_2) \rightarrow 1 + z^2 - \lambda_1(z^2 + xy) - \lambda_2(x^2 + y^2 - 1)$$

$$\begin{cases} D_1 L = -y\lambda_1 - 2x\lambda_2 = 0 \\ D_2 L = -x\lambda_1 - 2y\lambda_2 = 0 \\ D_3 L = 2z - 2z\lambda_1 = 0 \\ D_4 L = -z^2 - xy = 0 \\ D_5 L = -x^2 - y^2 + 1 = 0 \end{cases} \stackrel{(z=0)}{\Leftrightarrow} \begin{cases} -y\lambda_1 - 2x\lambda_2 = 0 \\ -x\lambda_1 - 2y\lambda_2 = 0 \\ 0 = 0 \\ xy = 0 \\ x^2 + y^2 = 1 \end{cases}$$

merk op dat door vgl 5  $x$  en  $y$   
 niet tegelijk 0 kunnen zijn.

$$\begin{aligned} (x=0) \\ \Leftrightarrow \end{aligned} \left\{ \begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = 0 \\ x = 0 \\ y = 1 \end{array} \right. \quad \vee \quad \left\{ \begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = 0 \\ x = 0 \\ y = -1 \end{array} \right.$$

$$\begin{aligned} (y=0) \\ \Leftrightarrow \end{aligned} \left\{ \begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = 0 \\ y = 0 \\ x = 1 \end{array} \right. \quad \vee \quad \left\{ \begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = 0 \\ y = 0 \\ x = -1 \end{array} \right.$$

$$\begin{aligned} (z \neq 0) \\ \Leftrightarrow \end{aligned} \left\{ \begin{array}{l} \lambda_2 = y/2x \\ \lambda_2 = x/2y \\ \lambda_1 = 1 \\ -z^2 - xy = 0 \\ -x^2 - y^2 + 1 = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} y^2 = x^2 \\ \lambda_2 = x/2y \\ \lambda_1 = 1 \\ -z^2 - xy = 0 \\ x = \pm \sqrt{\frac{1}{z}} \end{array} \right.$$

merk op dat als  $x$  of  $y$  nul is dat dan ook de ander nul moet zijn. Daarom mogen we delen door  $x$  en  $y$ .

Merk bovendien op dat door vgl 4  $x$  en  $y$  een verschillend teken moeten hebben.

$$\begin{aligned} (x = \sqrt{\frac{1}{z}}) \\ \Leftrightarrow \end{aligned} \left\{ \begin{array}{l} y = -\sqrt{\frac{1}{z}} \\ \lambda_2 = -1/2 \\ \lambda_1 = 1 \\ z = \sqrt{\frac{1}{z}} \end{array} \right. \quad \vee \quad \left\{ \begin{array}{l} y = -\sqrt{\frac{1}{z}} \\ \lambda_2 = -1/2 \\ \lambda_1 = 1 \\ z = -\sqrt{\frac{1}{z}} \end{array} \right.$$

$$\begin{aligned} (x = -\sqrt{\frac{1}{z}}) \\ \Leftrightarrow \end{aligned} \left\{ \begin{array}{l} y = \sqrt{\frac{1}{z}} \\ \lambda_2 = -1/2 \\ \lambda_1 = 1 \\ z = \sqrt{\frac{1}{z}} \end{array} \right. \quad \vee \quad \left\{ \begin{array}{l} y = \sqrt{\frac{1}{z}} \\ \lambda_2 = -1/2 \\ \lambda_1 = 1 \\ z = -\sqrt{\frac{1}{z}} \end{array} \right.$$

De 8 kritieke punten zijn dus:

$$(0, 1, 0) \vee (0, -1, 0) \vee (1, 0, 0) \vee (-1, 0, 0) \vee (1/\sqrt{2}, -1/\sqrt{2}, -1/\sqrt{2}) \\ (1/\sqrt{2}, -1/\sqrt{2}, 1/\sqrt{2}) \vee (-1/\sqrt{2}, 1/\sqrt{2}, -1/\sqrt{2}) \vee (-1/\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2})$$

Merck op dat

$$\begin{aligned} f(0,1,0) &= f(0,-1,0) = f(1,0,0) = f(-1,0,0) = 1 \\ f(1/\sqrt{2}, -1/\sqrt{2}, 1/\sqrt{2}) &= f(1/\sqrt{2}, -1/\sqrt{2}, -1/\sqrt{2}) = f(-1/\sqrt{2}, 1/\sqrt{2}, -1/\sqrt{2}) \\ &= f(-1/\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2}) = \frac{3}{2} \end{aligned}$$

Omdat we op zoek zijn naar een minimum, komen de laatste 4 punten niet in aanmerking. Bovendien moeten we maar van 1 via vier andere punten bewijzen dat het een minimum is.

$$\text{Hessiaan} = \begin{pmatrix} -2\lambda_2 & -\lambda_1 & 0 & -y & -2x \\ -\lambda_1 & -2\lambda_2 & 0 & -x & -2y \\ 0 & 0 & 2-2\lambda_1 & -2z & 0 \\ -y & -x & -2z & 0 & 0 \\ -2x & -2y & 0 & 0 & 0 \end{pmatrix}$$

$$H(1,0,0,0,0) = \begin{pmatrix} 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Delta_1 = -2(1 \cdot 2 \cdot 1 \cdot (-2)) = 8$$

hetzelfde teken als  $(-1)^m$  dus minimum.

5.  $W: \mathbb{R}^2 \rightarrow \mathbb{R}: (R_1, R_2) \rightarrow 200(50 + 3\sqrt{R_1}) + 200(70 + 2\sqrt{R_2}) - 100$   
onder de RV:  $R_1 + R_2 = 100.000$

Substitutie van de RV in de winstfunctie geeft:

$$W: \mathbb{R} \rightarrow \mathbb{R}: R \rightarrow 200(120 + 3\sqrt{100.000 - R}) + 200\sqrt{R} - 100$$

$$W'(R) = 0$$

$$\Leftrightarrow 300 \cdot (-1) \cdot (100.000 - R)^{-1/2} + 200 \cdot R^{-1/2} = 0$$

$$\Leftrightarrow -300 + 200 \frac{\sqrt{100.000 - R}}{\sqrt{R}} = 0$$

$$\Leftrightarrow \frac{\sqrt{100.000 - R}}{R} = 1,5$$

$$\Leftrightarrow 100.000 - R = 2,25R$$

$$\Leftrightarrow R = 30769,23$$

maximum?

$$W''(30769,23) = -150 \cdot (100000 - R)^{-3/2} - 100 \cdot R^{-3/2} < 0$$

→ maximum

$$R_1 = 100000 - R = 69.230,77$$

$$R_2 = R = 30.769,23$$

6. minimaliseer  $4x_1 + 5x_2 + 7x_3$

Onder de RV  $x_1^{0,3} \cdot x_2^{0,5} \cdot x_3^{0,2} = 20$

$$\nabla g = (0,0,0) \Leftrightarrow (0,3x_1^{-0,7}x_2^{0,5}x_3^{0,2}, 0,5x_1^{0,3}x_2^{-0,5}x_3^{0,2}, 0,2x_1^{0,3}x_2^{0,5}x_3^{-0,8}) = (0,0,0)$$

$$\Leftrightarrow x_1 = x_2 = x_3 = 0$$

→ voldoet niet aan RV

De Lagrangefunctie is:

$$L: \mathbb{R}^4 \rightarrow \mathbb{R}: (x_1, x_2, x_3, \lambda) \rightarrow 4x_1 + 5x_2 + 7x_3 - \lambda(x_1^{0,3}x_2^{0,5}x_3^{0,2} - 20)$$

$$\begin{cases} D_1 L = 4 - 0,3\lambda x_1^{-0,7}x_2^{0,5}x_3^{0,2} = 0 \\ D_2 L = 5 - 0,5\lambda x_1^{0,3}x_2^{-0,5}x_3^{0,2} = 0 \\ D_3 L = 7 - 0,2\lambda x_1^{0,3}x_2^{0,5}x_3^{-0,8} = 0 \\ D_4 L = -x_1^{0,3}x_2^{0,5}x_3^{0,2} + 20 = 0 \end{cases} \quad \begin{array}{l} \text{merk op dat } x_1 \neq 0 \text{ en} \\ x_2 \neq 0 \text{ en } x_3 \neq 0 \text{ en } \lambda \neq 0 \end{array}$$

$$\Leftrightarrow \begin{cases} \lambda = \frac{40}{3} \cdot x_1^{0,7}x_2^{-0,5}x_3^{-0,2} \\ \lambda = 10 \cdot x_1^{-0,3}x_2^{0,5}x_3^{-0,2} \\ \lambda = 35 \cdot x_1^{-0,3}x_2^{-0,5}x_3^{0,8} \\ x_1^{0,3}x_2^{0,5}x_3^{0,2} = 20 \end{cases} \quad \begin{cases} \frac{4}{3}x_1 \cdot x_2^{-1} = 1 \\ 3,5 \cdot x_2^{-1} \cdot x_3 = 1 \\ \lambda = 35x_1^{-0,3}x_2^{-0,5}x_3^{0,8} \\ x_1^{0,3}x_2^{0,5}x_3^{0,2} = 20 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 = \frac{3}{4}x_2 \\ x_3 = \frac{2}{7}x_2 \\ \lambda = 35x_1^{-0,3}x_2^{-0,5}x_3^{0,8} \\ \left(\frac{3}{4}x_2\right)^{0,3}x_2^{0,5}\left(\frac{2}{7}x_2\right)^{0,2} = 20 \end{cases} \quad \begin{cases} x_1 = 21 \\ x_3 = 8 \\ \lambda = 14 \\ x_2 = 28 \end{cases}$$

is dit een minimum?

$$H(21, 28, 8, 14) = \begin{pmatrix} 0,133 & -0,071 & -0,1 & -0,286 \\ -0,071 & 0,089 & -0,125 & -0,357 \\ -0,1 & -0,125 & 0,7 & -0,5 \\ -0,286 & -0,357 & -0,5 & 0 \end{pmatrix}$$

$$\Delta_1 = -0,042 < 0 \quad \Delta_2 = -0,156$$

zelfde teken als  $(-1)^m$  dus minimum

7. a) maximaliseer de functie

$$W: \mathbb{R}^4 \rightarrow \mathbb{R}: (R_1, P_1, R_2, P_2) \rightarrow \left( \frac{160R_1}{160+R_1} + \frac{320P_1}{80+P_1} \right) \cdot 400 + \left( \frac{40R_2}{40+R_2} + \frac{120P_2}{30+P_2} \right) \cdot 900 - 100(R_1 + R_2 + P_1 + P_2)$$

$$\begin{cases} D_1 W = 0 = (160(160+R_1) - 160R_1) / (160+R_1)^2 \cdot 400 - 100 \\ D_2 W = 0 = (320(80+P_1) - 320P_1) / (80+P_1)^2 \cdot 400 - 100 \\ D_3 W = 0 = (40(40+R_2) - 40R_2) / (40+R_2)^2 \cdot 900 - 100 \\ D_4 W = 0 = (120(30+P_2) - 120P_2) / (30+P_2)^2 \cdot 900 - 100 \end{cases}$$

$$\Leftrightarrow \begin{cases} 0,25 (160+R_1)^2 = 160^2 \\ 0,25 (80+P_1)^2 = 320 \cdot 80 \\ 119 \cdot (40+R_2)^2 = 40^2 \\ 119 \cdot (30+P_2)^2 = 120 \cdot 30 \end{cases} \quad \begin{cases} (160+R_1)^2 = 320^2 \\ (80+P_1)^2 = 320^2 \\ (40+R_2)^2 = 120^2 \\ (30+P_2)^2 = 180^2 \end{cases}$$

$$\begin{cases} R_1 = 160 \\ P_1 = 240 \\ R_2 = 80 \\ P_2 = 150 \end{cases}$$

$$\text{Hessiaan} = \begin{pmatrix} -0,625 & 0 & 0 & 0 \\ 0 & -0,625 & 0 & 0 \\ 0 & 0 & -1,666 & 0 \\ 0 & 0 & 0 & -1,111 \end{pmatrix}$$

alle eigenwaarden zijn negatief  $\rightarrow$  negatief definitief  
 $\rightarrow$  maximum

b) maximaliseer de functie W

onder de RV  $R_1 + P_1 + R_2 + P_2 = 480$

$$\nabla g = (1, 1, 1, 1) \neq (0, 0, 0, 0)$$

de Lagrangefunctie is:

$$L: \mathbb{R}^5 \rightarrow \mathbb{R}: (R_1, P_1, R_2, P_2, \lambda) \rightarrow \left( \frac{160R_1}{160+R_1} + \frac{320P_1}{80+P_1} \right) \cdot 400 + \left( \frac{40R_2}{40+R_2} + \frac{120P_2}{30+P_2} \right) \cdot 900 - 100(R_1 + R_2 + P_1 + P_2) - \lambda(R_1 + R_2 + P_1 + P_2 - 480)$$

$$\begin{cases} D_1 L = 0 = (160(160+R_1) - 160R_1) / (160+R_1)^2 \cdot 400 - 100 - \lambda \\ D_2 L = 0 = (320(80+P_1) - 320P_1) / (80+P_1)^2 \cdot 400 - 100 - \lambda \\ D_3 L = 0 = (40(40+R_2) - 40R_2) / (40+R_2)^2 \cdot 900 - 100 - \lambda \\ D_4 L = 0 = (120(30+P_2) - 120P_2) / (30+P_2)^2 \cdot 900 - 100 - \lambda \\ D_5 L = 0 = -R_1 - P_1 - R_2 - P_2 + 480 \end{cases}$$

$$\begin{cases} (160 + R_1)^2 = 160^2 \cdot 20^2 / \sqrt{100 + \lambda}^2 \\ (80 + P_1)^2 = 160^2 \cdot 20^2 / \sqrt{100 + \lambda}^2 \\ (40 + R_2)^2 = 40^2 \cdot 30^2 / \sqrt{100 + \lambda}^2 \\ (30 + P_2)^2 = 60^2 \cdot 30^2 / \sqrt{100 + \lambda}^2 \\ R_1 + R_2 + P_1 + P_2 = 480 \end{cases}$$

$$\begin{cases} (160 + R_1) \sqrt{100 + \lambda} = 3200 \\ (80 + P_1) \sqrt{100 + \lambda} = 3200 \\ (40 + R_2) \sqrt{100 + \lambda} = 1200 \\ (30 + P_2) \sqrt{100 + \lambda} = 1800 \\ R_1 + R_2 + P_1 + P_2 = 480 \quad (*) \end{cases}$$

$$\begin{cases} R_1 = (3200 / \sqrt{100 + \lambda}) - 160 \\ P_1 = (3200 / \sqrt{100 + \lambda}) - 80 \\ R_2 = (1200 / \sqrt{100 + \lambda}) - 40 \\ P_2 = (1800 / \sqrt{100 + \lambda}) - 30 \\ (9400 / \sqrt{100 + \lambda}) = 790 \end{cases}$$

$$\begin{cases} R_1 = 108,93 \\ P_1 = 188,94 \\ R_2 = 60,85 \\ P_2 = 121,28 \\ \lambda = 41,58 \end{cases} \quad \text{Hessiaan} = \begin{pmatrix} -1,052 & 0 & 0 & 0 & -1 \\ 0 & -1,052 & 0 & 0 & -1 \\ 0 & 0 & -2,808 & 0 & -1 \\ 0 & 0 & 0 & -1,872 & -1 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix}$$

$$\Delta_3 = 4,68$$

$$\Delta_2 = -10,18$$

$$\Delta_1 = -16,239$$

tekens alterneren te beginnen met het teken van  $(-1)^m \rightarrow$  maximum.

$$W = 176.151,90$$

c) Schatting:  $176.151,90 + 41,58 = 176.193,48$

werkelijk: In vgl (\*) wordt 480 nu 481 dan voegt.

$$\begin{cases} R_1 = 5136/47 \\ P_1 = 8896/47 \\ R_2 = 2866/47 \\ P_2 = 5709/47 \\ \lambda = 41,22 \end{cases} \rightarrow \text{de winst wordt dan } W = 176.193,30$$

d)  $B = 160 + 240 + 80 + 150 = 630$

in vgl (x) wordt  $480 = 630$ . dan volgt:

$$\begin{cases} R_1 = 160 \\ P_1 = 240 \\ R_2 = 80 \\ P_2 = 150 \\ \lambda = 0 \end{cases}$$

e) De Nieuwe Lagrangefunctie wordt:

$$L: \mathbb{R}^6 \rightarrow \mathbb{R}: (R_1, P_1, R_2, P_2, \lambda_1, \lambda_2) \rightarrow \left( \frac{160R_1 + 320P_1}{160+R_1} \right) \cdot 400 + \left( \frac{40R_2 + 120P_2}{40+R_2} \right) \cdot 900 - 100(R_1+R_2+P_1+P_2) - \lambda_1(R_1+R_2-200) - \lambda_2(P_1+P_2-280)$$

Het bijbehorende stelsel, na invullen in vorige berekeningen is:

$$\begin{cases} R_1 = (3200 / \sqrt{100+\lambda_1}) - 160 \\ P_1 = (3200 / \sqrt{100+\lambda_1}) - 80 \\ R_2 = (1200 / \sqrt{100+\lambda_2}) - 40 \\ P_2 = (1800 / \sqrt{100+\lambda_2}) - 30 \\ (4400 / \sqrt{100+\lambda_1}) = 400 \\ (5000 / \sqrt{100+\lambda_2}) = 390 \end{cases} \Leftrightarrow \begin{cases} R_1 = 1440/11 \\ P_1 = 848/5 \\ R_2 = 760/11 \\ P_2 = 552/5 \\ \lambda_1 = 21 \\ \lambda_2 = 64,366 \end{cases}$$

De winst die dan gemaakt wordt is  $W = 175.497,44$

f) Promotie want de Lagrangemultipliator is groter.

8. maximaliseer  $8xyz$   
 onder de RV  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$\nabla g = (0,0,0) \rightarrow (\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}) = (0,0,0)$

$\rightarrow x = y = z = 0 \rightarrow$  niet in RV

De Lagrangefunctie is:

$L: \mathbb{R}^4 \rightarrow \mathbb{R}: (x,y,z,\lambda) \rightarrow 8xyz - \lambda (\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1)$

$$\begin{cases} D_1 L = 0 = 8yz - 2\lambda x / a^2 \\ D_2 L = 0 = 8xz - 2y\lambda / b^2 \\ D_3 L = 0 = 8xy - 2z\lambda / c^2 \\ D_4 L = 0 = -x^2/a^2 - y^2/b^2 - z^2/c^2 + 1 \end{cases} \begin{cases} \lambda = 4yz a^2 / x \\ \lambda = 4xz b^2 / y \\ \lambda = 4xy c^2 / z \\ x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \end{cases}$$

$$\begin{cases} x^2 b^2 = y^2 a^2 \\ x^2 c^2 = z^2 a^2 \\ \dots \\ \dots \end{cases} \begin{cases} y = x \cdot b / a \\ z = x \cdot c / a \\ \lambda = 4yz a^2 / x \\ \frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1 \end{cases} \begin{cases} y = b / \sqrt{3} \\ z = c / \sqrt{3} \\ \lambda = 4abc / \sqrt{3} \\ x = \frac{a}{\sqrt{3}} \end{cases}$$

is dit een maximum?

Hessiaan =

$$\begin{pmatrix} -2\lambda/a^2 & 8Z & 8Y & -2\lambda/a^2 \\ 8Z & -2\lambda/b^2 & 8X & -2\lambda/b^2 \\ 8Y & 8X & -2\lambda/c^2 & -2\lambda/c^2 \\ -2\lambda/a^2 & -2\lambda/b^2 & -2\lambda/c^2 & 0 \end{pmatrix}$$

$$H\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}, \frac{4abc}{\sqrt{3}}\right) = \begin{pmatrix} -8bc/a\sqrt{3} & 8c/\sqrt{3} & 8b/\sqrt{3} & -2/a\sqrt{3} \\ 8c/\sqrt{3} & -8ac/b\sqrt{3} & 8a/\sqrt{3} & -2/b\sqrt{3} \\ 8b/\sqrt{3} & 8a/\sqrt{3} & -8ab/c\sqrt{3} & -2/c\sqrt{3} \\ -2/a\sqrt{3} & -2/b\sqrt{3} & -2/c\sqrt{3} & 0 \end{pmatrix}$$

$$\Delta_2 = \frac{1}{3\sqrt{3}} \cdot \left( \frac{-2}{b} \cdot \left( \frac{-16a}{c} - \frac{16a}{c} \right) + \frac{2}{c} \left( \frac{16a}{b} + \frac{16a}{b} \right) \right)$$

$$= \frac{1}{3\sqrt{3}} \cdot \left( \frac{64a}{bc} + \frac{64a}{bc} \right) = \frac{128a}{3\sqrt{3}bc} > 0$$

$$\Delta_1 = \frac{1}{9} \left[ \frac{-8bc}{a} \cdot \left( \frac{-8ac}{b} \left( \frac{-4}{c^2} \right) - 8a \left( \frac{-4}{bc} \right) - \frac{2}{b} \left( \frac{-16a}{c} - \frac{16a}{c} \right) \right) \right.$$

$$- 8c \cdot \left( 8c \cdot \left( \frac{-4}{c^2} \right) - 8a \left( \frac{-4}{ac} \right) - \frac{2}{b} \cdot \left( \frac{-16b}{c} - \frac{16b}{c} \right) \right)$$

$$+ 8b \cdot \left( 8c \left( \frac{-4}{bc} \right) + \frac{8ac}{b} \left( \frac{-4}{ac} \right) - \frac{2}{b} (-16 + 16) \right)$$

$$+ \frac{2}{a} \left( 8c \left( \frac{-16a}{c} - \frac{16a}{c} \right) + \frac{8ac}{b} \left( \frac{-16b}{c} - \frac{16b}{c} \right) + 8a(-16 + 16) \right)$$

$$= \frac{1}{9} \left[ -256 - 256 - 512 + 256 - 256 - 512 - 256 - 256 - 512 - 512 \right]$$

$$= \frac{-1024}{3} < 0$$

teken is alternerend te beginnen met het teken van  $(-1)^{m+1}$   
 $\rightarrow$  maximum.

9. a) F heeft een lokaal extremum:  $F'(x^*) = 0$

$$\Leftrightarrow F'(x^*) = D_1 F(x^*, h(x^*)) + D_2 F(x^*, h(x^*)) \cdot h'(x^*) = 0 \quad (1)$$

$$\text{Nu is } h'(x) + e^{h(x)} + h'(x) \cdot x e^{h(x)} = 0$$

$$\Leftrightarrow (1 + x e^{h(x)}) \cdot h'(x) = -e^{h(x)}$$

$$\Leftrightarrow h'(x) = \frac{-e^{h(x)}}{1 + x e^{h(x)}}$$

invullen in (1) geeft

$$D_1 F(x^*, h(x^*)) = -D_2 F(x^*, h(x^*)) \cdot \frac{-e^{h(x^*)}}{1 + x e^{h(x^*)}}$$

$$\Leftrightarrow (x^* + e^{-h(x^*)}) D_1 F(x^*, h(x^*)) = D_2 F(x^*, h(x^*))$$

b) Definieer de functie  $Z: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, y) \rightarrow y + xe^y$   
 en een  $c = 3$ .

voordat we de IFS kunnen toepassen moet er aan drie voorwaarden voldaan zijn.

- 1)  $Z$  heeft continue PA: check
- 2)  $\exists (x^*, y^*) \in \mathbb{R}^2 : Z(x^*, y^*) = c$   
 $Z(0, 3) = 3$
- 3)  $D_2 Z(0, 3) = 1 \neq 0$

Dan geldt volgens de IFS dat er een open interval  $I \subseteq \mathbb{R}$  bestaat met  $0 \in I$  en een open  $J \subseteq \mathbb{R}$  bestaat met  $3 \in J$  zodat voor alle  $x \in I$  en  $y \in J$  geldt dat  $D_2 Z(x, y) \neq 0$  en zodat er een unieke functie  $h: I \rightarrow J$  die voldoet aan 1)  $h(0) = 3$

2)  $\forall x \in I : Z(x, h(x)) = 3$   
 of nog  $h(x) + xe^{h(x)} = 3$  voor alle  $x \in I$

In de buurt van 0 kunnen we dus  $f(x, y)$  vervangen door  $f(x, h(x)) = F(x)$ . De nodige voorwaarde is dus dezelfde als in a).

10. a)  $D_1 h(x, y, z) = f'(xy+z^2) \cdot y$   
 $D_2 h(x, y, z) = f'(xy+z^2) \cdot x$   
 $D_3 h(x, y, z) = f'(xy+z^2) \cdot 2z$

voor  $x=y=z=0$  geldt dan:

$D_1 h = D_2 h = D_3 h = 0 \rightarrow$  kritiek punt.

$$H(0,0,0) = \begin{pmatrix} 0 & f'(0) & 0 \\ f'(0) & 0 & 0 \\ 0 & 0 & 2f'(0) \end{pmatrix} \quad \det = -f'(0) \cdot 2f'(0)^2$$

$$= -2f'(0)^3 < 0$$

$$\det H_2 = -f'(0)^2$$

$$\det H_1 = 0$$

Omdat de determinanten niet het strikte verloop hebben van het schema en omdat  $\det H_3 \neq 0$  zal er minstens 1 positieve (strikt) en 1 negatieve (strikt) eigenwaarde zijn  $\rightarrow$  geen extremum.

b) 1. Lagrangemethode.

$L: \mathbb{R}^4 \rightarrow \mathbb{R}: (x, y, z, \lambda) \rightarrow f(xy+z^2) - \lambda(xe^y - y \cos x)$

$$\begin{cases} D_1 L = 0 = f'(xy+z^2) \cdot y - \lambda e^y + y \lambda \sin x \\ D_2 L = 0 = f'(xy+z^2) \cdot x - \lambda x e^y + \lambda \cos x \\ D_3 L = 0 = f'(xy+z^2) \cdot 2z \\ D_4 L = 0 = -x e^y + y \cos x \end{cases}$$

Merk op dat voor  $x=y=z=\lambda=0$  de vgl kloppen.  
 is dit een minimum?

$$H(0,0,0,0) = \begin{pmatrix} 0 & f'(0) & 0 & -1 \\ f'(0) & 0 & 0 & 1 \\ 0 & 0 & 2f'(0) & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$

$\Delta_2 = -2f'(0) < 0$

$\Delta_1 = -f'(0) \cdot (f'(0) \cdot 0 + 1 \cdot 2f'(0)) + 1 \cdot f'(0) \cdot -2f'(0)$   
 $= -2f'(0)^2 - 2f'(0)^2$   
 $= -4f'(0)^2 < 0$

zoude teken als  $(-1)^m$  dus minimum

## 2. impliciete substitutie

Beschouw de functie  $F: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, y) \rightarrow xe^y - y \cos x$   
en een  $c = 0$

Voordat we de IFS kunnen toepassen moeten er drie voorwaarden voldaan zijn:

- 1)  $F$  heeft continue PA: check
- 2)  $F(0, 0) = 0$
- 3)  $D_2 F(0, 0) = -1 \neq 0$

Dan geldt volgens de IFS dat er een open  $A \subseteq \mathbb{R}$  bestaat met  $0 \in A$  en een open  $B \subseteq \mathbb{R}$  bestaat met  $0 \in B$  zodat  $D_2 F(x, y) \neq 0$  voor alle  $x \in A$  en  $y \in B$  en zodat er een unieke functie  $t: A \rightarrow B$  bestaat die voldoet aan:

- 1)  $t(0) = 0$
- 2)  $\forall x \in A: F(x, t(x)) = 0$   
of nog  $\forall x \in A: xe^{t(x)} - t(x) \cos x = 0$

In de buurt van  $(0, 0)$  kunnen we  $y$  dus vervangen door  $t(x)$ :

Minimaliseer  $h(x, t(x), z) = f(x, t(x), z^2)$

Is  $(0, t(0), 0)$  een kritiek punt voor  $h$ ?

Definieer de functie  $k: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, z) \rightarrow h(x, t(x), z)$

is  $(0, 0)$  een kritiek punt voor  $k$ ?

$$D_1 k(x, z) = f'(x, t(x) + z^2) \cdot (t'(x) + x \cdot t''(x)) \stackrel{!}{=} 0$$

$$D_2 k(x, z) = f'(x, t(x) + z^2) \cdot 2z \stackrel{!}{=} 0$$

$(0, 0)$  is duidelijk een oplossing van dit stelsel  
is dit nu een minimum?

$$\text{Hessiaan}(0, 0) = \begin{pmatrix} f'(0) \cdot t'(0) + f''(0) \cdot t'(0) & 0 \\ 0 & 2f'(0) \end{pmatrix}$$

Nu geldt:  $xe^{t(x)} - t(x) \cos x = 0 \quad \forall x \in A$

$$\text{vullen we 0 in: } e^{t(x)} + xe^{t(x)} \cdot t'(x) - \cos x \cdot t'(x) + t(x) \cdot \sin x = 0$$

$$1 + 0 - t'(0) + 0 = 0$$

$$t'(0) = 1$$

invullen in de Hessiaan wordt aan:

$$H_{(0,0)} = \begin{pmatrix} 2f'(0) & 0 \\ 0 & 2f'(0) \end{pmatrix}$$

$$\det = 4f'(0)^2 > 0$$

en  $2f'(0) < 0$  dus minimum

### 3.3.6. opdrachten

---

1. a) JUUST als een functie afleidbaar is, is ze ook continu. de functie is dus integreerbaar.

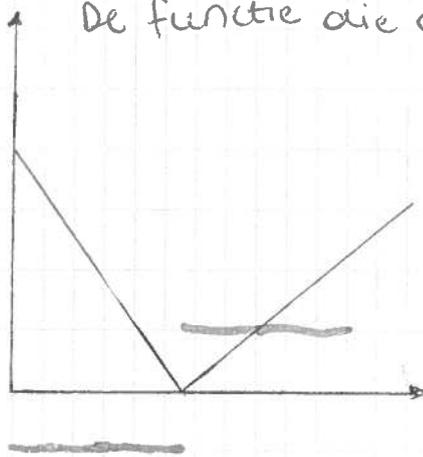
b) JUUST bv.  $f: \mathbb{R} \rightarrow \mathbb{R}: x \rightarrow \begin{cases} 1 & \text{als } x > 0 \\ 0 & \text{als } x \leq 0 \end{cases}$

c) FOUT

d) JUUST. Als we kunnen bewijzen dat  $g$  afleidbaar is, is bewezen dat ze continu is.

Nu is  $g'(x) = f(x)$ .

2. een functie die in 1 punt niet afleidbaar is, is bv. De functie die deze als integraal heeft is: \*



3. zij  $F$  een primitieve functie van  $f$ .

a)  $g'(x) = (F(x^2) - F(1))' = 2x \cdot f(x^2)$

b)  $h'(x) = (F(t+1) - F(t))' = f(t+1) - f(t)$

4.  $g'(x) = (F(x^3) \cdot F(0))' = 3x^2 \cdot f(x^3) \cdot F(0)$

5. a) Juust

b) Juust

c) Juust want  $f(1) = 0 \Rightarrow g'(1) = 0$

d)  $g''(x) = f'(x) > 0 \rightarrow$  FOUT

e)  $g''(1) = f'(1) > 0$  en  $g'(1) = f(1) = 0 \rightarrow$  JUIST

f) FOUT, zie d

g) JUIST

$$6. W: \mathbb{R} \rightarrow \mathbb{R}^+ : t \rightarrow W(t) = W_0 + \int_0^t D(x) dx - \int_0^t A(p) dp$$

extremaal:  $W'(t) = 0$

$$\Leftrightarrow D(t) - A(t) = 0$$

$$\Leftrightarrow D(t) = A(t)$$

$$W''(t) = D'(t) - A'(t)$$

ALS  $D'(t) > A'(t)$  dan minimum

ALS  $D'(t) < A'(t)$  dan maximum

$$7. G: \mathbb{R}^+ \rightarrow \mathbb{R} : t \rightarrow G(t) = \frac{\int_0^t S(p) dp}{t}$$

$$b) G'(t) = 0$$

$$\Leftrightarrow \frac{S(t)}{t} - \frac{\int_0^t S(p) dp}{t^2} = 0$$

$$\Leftrightarrow S(t) = G(t)$$

### 3.4.4 oplossen

$$1. a) \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = - \int \frac{d(\cos x)}{\cos x} = -\ln |\cos x| + C$$

$$b) \int \frac{x}{x^4 + 2x^2 + 2} \, dx = \int \frac{x}{(x^2+1)^2 + 1} \, dx = \frac{1}{2} \int \frac{d(x^2+1)}{(x^2+1)^2 + 1}$$

$$= \frac{1}{2} \operatorname{Bgtan} |x^2+1| + C$$

$$c) \int \frac{dx}{\sqrt{-(x-2)^2 + 1}} = \operatorname{Bgsin} |x-2| + C$$

$$d) \int \frac{x \, dx}{x^4 + 1} = \int \frac{x \, dx}{x^2^2 + 1^2} = \frac{1}{2} \int \frac{d(x^2)}{(x^2)^2 + 1} = \frac{1}{2} \operatorname{Bgtan} |x^2| + C$$

$$2. a) \int \operatorname{Bgtan} x \, dx \quad \text{stel } u = x \text{ en } v = \operatorname{Bgtan} x$$

dan is  $u' = 1$  en  $v' = 1/(1+x^2)$

$$= x \operatorname{Bgtan} x - \int \frac{x}{1+x^2} \, dx$$

$$= x \operatorname{Bgtan} x - \frac{1}{2} \int \frac{d(1+x^2)}{1+x^2} = x \operatorname{Bgtan} x - \frac{1}{2} \ln |1+x^2| + C$$

$$b) \int \operatorname{Bgsin} x \, dx \quad \text{stel } u = x \text{ en } v = \operatorname{Bgsin} x$$

dan is  $u' = 1$  en  $v' = 1/\sqrt{1-x^2}$

$$= x \operatorname{Bgsin} x - \int \frac{x}{\sqrt{1-x^2}} \, dx = x \operatorname{Bgsin} x + \frac{1}{2} \int \frac{d(1-x^2)}{\sqrt{1-x^2}}$$

$$= x \operatorname{Bgsin} x + \frac{1}{2} \cdot \sqrt{1-x^2} \cdot 2 = x \operatorname{Bgsin} x + \sqrt{1-x^2} + C$$

$$c) \int \ln |x| \, dx \quad \text{stel } u = x \text{ en } v = \ln |x|$$

dan is  $u' = 1$  en  $v' = 1/x$

$$= x \ln |x| - \int dx = x \ln |x| - x + C$$

$$d) \int \sqrt{1-x^2} \, dx \quad \text{stel } u = x \text{ en } v = \sqrt{1-x^2}$$

dan is  $u' = 1$  en  $v' = -x/\sqrt{1-x^2}$

$$= x \sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}} \, dx = x \sqrt{1-x^2} - \int \sqrt{1-x^2} \, dx + \operatorname{Bgsin} x$$

$$\Leftrightarrow \int \sqrt{1-x^2} \, dx = \frac{1}{2} (x \sqrt{1-x^2} + \operatorname{Bgsin} x)$$

$$\begin{aligned}
 \text{e) } \int \sin x e^x dx & \text{ stel } u = e^x \text{ en } v = \sin x \\
 & \text{ dan is } u' = e^x \text{ en } v' = \cos x \\
 & = e^x \sin x - \int \cos x e^x dx \text{ stel } u = e^x \text{ en } v = \cos x \\
 & \text{ dan is } u' = e^x \text{ en } v' = -\sin x \\
 & = e^x \sin x - e^x \cos x - \int \sin x e^x dx \\
 \Leftrightarrow 2 \int \sin x e^x dx & = e^x \sin x - e^x \cos x \\
 \Leftrightarrow \int \sin x e^x dx & = \frac{1}{2} e^x (\sin x - \cos x) + C
 \end{aligned}$$

$$\begin{aligned}
 \text{f) } \int \cos x e^x dx & \text{ stel } u = e^x \text{ en } v = \cos x \\
 & \text{ dan is } u' = e^x \text{ en } v' = -\sin x \\
 & = \cos x e^x + \int \sin x e^x dx = \cos x e^x + \frac{1}{2} e^x (\sin x - \cos x) + C \\
 & = \frac{1}{2} e^x (\sin x + \cos x) + C
 \end{aligned}$$

$$\begin{aligned}
 3. \text{ a) } \int a^x dx & = \int e^{x \ln a} dx = \int e^{x \ln a} dx \\
 & = \frac{e^{x \ln a} d(e^{x \ln a})}{\ln a} = \frac{e^{x \ln a}}{\ln a} = \frac{a^x}{\ln a}
 \end{aligned}$$

$$\text{b) } \int \frac{4}{x^2 - 4x + 4} dx = \int \frac{4}{(x-2)^2} dx = \frac{-4}{x-2} + C$$

$$\begin{aligned}
 \text{c) } \int \frac{dx}{x^2 - 4x + 7} & = \int \frac{dx}{(x-2)^2 + 3} = \frac{1}{\sqrt{3}} \text{Bgtan} \frac{x-2}{\sqrt{3}} + C \\
 & = \frac{\sqrt{3}}{3} \text{Bgtan} \left( \frac{x-2}{\sqrt{3}} \right) + C
 \end{aligned}$$

$$\text{a) } \begin{array}{r} x^2 + x + 1 \\ - x^2 + 0x + 1 \\ \hline x \end{array} \left| \begin{array}{r} x^2 + 1 \\ 1 \end{array} \right.$$

$$\int \frac{x^2 + x + 1}{x^2 + 1} dx = \int dx + \int \frac{x}{x^2 + 1} = x + \frac{1}{2} \ln |x^2 + 1| + C$$

$$\text{e) } \int \frac{3x+3}{x^2+2x+1} dx = \frac{3}{2} \int \frac{2x+2}{x^2+2x+1} dx = \frac{3}{2} \ln |x^2+2x+1| + C$$

$$\begin{aligned}
 \text{f) } \int \frac{2x^2+3}{2x^2+1} dx & = \int dx + \int \frac{2}{2x^2+1} \\
 & = x + \int \frac{1}{x^2+1/2} \\
 & = x + \sqrt{2} \text{Bgtan} \sqrt{2}x + C
 \end{aligned}$$

$$\begin{aligned}
 g) \int \frac{\ln x \, dx}{x \ln x - x} & \text{ stel } y = \ln x \text{ dan is } dy = \frac{dx}{x} \\
 & = \int \frac{y \, dy}{y-1} = \int dy + \int \frac{1}{y-1} dy \\
 & = y + \ln |y-1| + C \\
 & = \ln |x| + \ln |\ln |y| - 1| + C \\
 & = \ln |x| \ln |y| - x| + C
 \end{aligned}$$

$$h) \int \frac{e^x dx}{1+e^{2x}} = \int \frac{d(e^x)}{1+e^{2x}} = \text{Bgtan}(e^x) + C$$

$$\begin{aligned}
 i) \int x \text{bgtan} x \, dx & \text{ stel } u = \frac{x^2}{2} \text{ en } v = \text{bgtan} x \\
 & \text{ dan is } u' = x \text{ en } v' = 1/(1+x^2) \\
 & = \frac{1}{2} x^2 \text{bgtan} x - \frac{1}{2} \int \frac{x^2}{1+x^2} \\
 & = \frac{1}{2} x^2 \text{bgtan} x - \frac{1}{2} \left( \int dx - \int \frac{1}{1+x^2} dx \right) \\
 & = \frac{1}{2} x^2 \text{bgtan} x - \frac{1}{2} x + \frac{1}{2} \text{bgtan} x + C
 \end{aligned}$$

$$\begin{aligned}
 j) \int x^3 e^{(x^2)} dx & \text{ stel } y = x^2 \text{ dan is } dy = 2x dx \\
 & = \frac{1}{2} \int y e^y \text{ stel } u = e^y \text{ en } v = y \\
 & \text{ dan is } u' = e^y \text{ en } v' = 1 \\
 & = \frac{1}{2} y e^y - \frac{1}{2} \int e^y = \frac{1}{2} (y e^y - e^y) + C \\
 & = \frac{1}{2} (x^2 e^{x^2} - e^{x^2}) + C
 \end{aligned}$$

$$\begin{aligned}
 k) \int e^{-x} \cos 3x \, dx & \text{ stel } u = -e^{-x} \text{ en } v = \cos 3x \\
 & \text{ dan is } u' = e^{-x} \text{ en } v' = -3 \sin 3x \\
 & = -e^{-x} \cos 3x - \int 3 \sin 3x e^{-x} \text{ stel } u = -e^{-x} \text{ en } v = \sin 3x \\
 & \text{ dan is } u' = e^{-x} \text{ en } v' = 3 \cos 3x \\
 & = -e^{-x} \cos 3x + 3 e^{-x} \sin 3x - 9 \int e^{-x} \cos 3x \, dx \\
 \Leftrightarrow \int e^{-x} \cos 3x \, dx & = \frac{e^{-x}}{10} (3 \sin 3x - \cos 3x)
 \end{aligned}$$

$$\begin{aligned}
 e) \int \ln^3(x) dx & \text{ stel } u=x \text{ en } v=\ln^3 x \\
 & \text{ dan is } u'=1 \text{ en } v'=3\ln^2 x \\
 & = x \ln^3(x) - \int 3\ln^2(x) dx \text{ stel } u=x \text{ en } v=\frac{x}{\ln^2(x)} \\
 & \text{ dan is } u'=1 \text{ en } v'=\frac{2\ln(x)}{x} \\
 & = x \ln^3(x) - 3x \ln^2(x) + 6 \int \ln(x) dx \text{ stel } u=x \text{ en } v=\ln(x) \\
 & \text{ dan is } u'=1 \text{ en } v'=\frac{1}{x} \\
 & = x \ln^3(x) - 3x \ln^2(x) + 6x \ln(x) - 6 \int dx \\
 & = x \ln^3(x) - 3x \ln^2(x) + 6x \ln(x) - 6x + C
 \end{aligned}$$

$$\begin{aligned}
 m) \int \sin(\ln(x)) dx & \text{ stel } u=x \text{ en } v=\sin(\ln(x)) \\
 & \text{ dan is } u'=1 \text{ en } v'=\frac{\cos(\ln(x))}{x} \\
 & = x \sin(\ln(x)) - \int \cos(\ln(x)) dx \\
 & \text{ stel } u=x \text{ en } v=\cos(\ln(x)) \\
 & \text{ dan is } u'=1 \text{ en } v'=-\frac{\sin(\ln(x))}{x} \\
 & = x \sin(\ln(x)) - x \cos(\ln(x)) - \int \sin(\ln(x)) dx \\
 \Rightarrow \int \sin(\ln(x)) dx & = \frac{1}{2} x \sin(\ln(x)) - \frac{1}{2} x \cos(\ln(x)) + C
 \end{aligned}$$

$$n) \int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x}{e^{2x} + 1} = \text{bgtan}(e^x) + C$$

$$o) \int \frac{\sin x}{1 + \cos x} dx = -\int \frac{d(1 + \cos x)}{1 + \cos x} = -\ln|1 + \cos x| + C$$

$$\begin{aligned}
 p) \int x(1+x)^{3/2} dx & \text{ stel } u=\frac{2}{5}(1+x)^{5/2} \text{ en } v=x \\
 & \text{ dan is } u'=(1+x)^{3/2} \text{ en } v'=1 \\
 & = \frac{2x}{5}(1+x)^{5/2} - \int \frac{2}{5}(1+x)^{5/2} dx \\
 & = \frac{2x}{5}(1+x)^{5/2} - \frac{4}{35}(1+x)^{7/2} + C
 \end{aligned}$$

$$q) \int \frac{e^x}{5(e^x - 1)} dx = \frac{1}{5} \int \frac{e^x}{e^x - 1} dx = \frac{1}{5} \ln|e^x - 1| + C$$

$$\begin{aligned}
 r) \int \frac{3x}{(x^2 - 1)^{3/2}} dx & \text{ stel } y=x^2 - 1 \text{ dan is } dy=2x dx \\
 & = \frac{3}{2} \int y^{-3/2} dy = \frac{3}{2} \frac{y^{-1/2}}{-1/2} = \frac{-3}{\sqrt{y}} = \frac{-3}{\sqrt{x^2 - 1}}
 \end{aligned}$$

4. a)  $\int_1^e \ln|x| dx$     stel  $u = x$  en  $v = \ln|x|$   
 dan is  $u' = 1$  en  $v' = 1/x$

$$= x \ln|x| \Big|_{x=1}^e - \int_1^e dx$$

$$= x \ln|x| - x \Big|_{x=1}^e = e - e - \ln|1| + 1 = 1$$

b)  $\int_0^4 \frac{dx}{1+\sqrt{x}}$     stel  $y = \sqrt{x}$  dan is  $dy = \frac{1}{2\sqrt{x}} dx$   
 $\Leftrightarrow dx = 2\sqrt{x} dy = 2y dy$

$$2 \int_0^2 \frac{y}{1+y} = 2 \int_0^2 \frac{y+1}{y+1} - 2 \int_0^2 \frac{1}{y+1}$$

$$= 2y - 2 \ln|y+1| \Big|_{x=0}^2$$

$$= 4 - 2 \ln|3| = 4 - 2 \ln|3|$$

c)  $\int_0^{\pi/2} \sin^2 x \cos x dx = \int_0^{\pi/2} \sin^2 x d(\sin x)$   
 $= \frac{\sin^3(x)}{3} \Big|_{x=0}^{\pi/2}$   
 $= 1/3 - 0 = 1/3$

d)  $\int_0^1 x^3 e^{-x^2} dx$     stel  $y = -x^2$  dan is  $dy = -2x dx$   
 $= \frac{1}{2} \int_0^{-1} y e^y dy$      $u = e^y$  en  $v = y$   
 $u' = e^y$  en  $v' = 1$

$$= \frac{1}{2} (y e^y - e^y) \Big|_{x=0}^{-1} = \frac{1}{2} (-x^2 \cdot e^{-x^2} - e^{-x^2}) \Big|_{x=0}^{-1}$$

$$= \frac{1}{2} (-1 e^{-1} - e^{-1} + 1)$$

$$= \frac{1}{2} - e^{-1} = \frac{1}{2} - \frac{1}{e}$$

e)  $\int_0^1 \frac{x}{(4+x^2)^2} dx$     stel  $y = 4+x^2$  dan is  $dy = 2x dx$

$$= \frac{1}{2} \int_4^5 \frac{1}{y^2} dy = \frac{-1}{2y} \Big|_{y=4}^5 = \frac{-1}{10} + \frac{1}{8} = \frac{1}{40}$$

f)  $\int_0^2 \frac{2x+1}{\sqrt{x+1}} dx = \int_0^2 \frac{2x+2}{\sqrt{x+1}} dx - \int_0^2 \frac{1}{\sqrt{x+1}}$

$$= 2 \int_0^2 \sqrt{x+1} - \int_0^2 (x+1)^{-1/2} = 2 \left( \frac{2}{3} (x+1)^{3/2} \right) - 2 (x+1)^{1/2} \Big|_{x=0}^2$$

$$= 4\sqrt{3} - 2\sqrt{3} - \frac{4}{3} + 2 = 2\sqrt{3} + \frac{2}{3}$$

$$g) x^2 - 3x + 2$$

$$D = 9 - 8 = 1$$

$$\frac{3 \pm 1}{2} = 1, 2$$

$$\int_0^3 |x^2 - 3x + 2| dx = \int_0^1 x^2 - 3x + 2 dx - \int_1^2 x^2 - 3x + 2 dx + \int_2^3 x^2 - 3x + 2 dx$$

$$= \left( \frac{x^3}{3} - \frac{3x^2}{2} + 2x \right) \Big|_{x=0}^1 - \left( \frac{x^3}{3} - \frac{3x^2}{2} + 2x \right) \Big|_{x=1}^2 + \left( \frac{x^3}{3} - \frac{3x^2}{2} + 2x \right) \Big|_{x=2}^3$$

$$= \frac{1}{3} - \frac{3}{2} + 2 - \frac{8}{3} + 6 - 4 + \frac{1}{3} - \frac{3}{2} + 2 + 9 - \frac{27}{2} + 6 - \frac{8}{3} + 6 - 4 = \frac{11}{6}$$

$$h) \int_0^2 (|x| + |1-x|) dx = \int_0^2 x dx + \int_0^2 |1-x| dx$$

$$= \int_0^2 x dx + \int_0^1 1-x dx - \int_1^2 1-x dx$$

$$= \frac{x^2}{2} \Big|_{x=0}^2 + x - \frac{x^2}{2} \Big|_{x=0}^1 - x + \frac{x^2}{2} \Big|_{x=1}^2$$

$$= 2 + 1 - \frac{1}{2} - 2 + 2 + 1 - \frac{1}{2} = 3$$

5. we weten dat  $2V_0 e^{R \cdot 0} = V_0 e^{R \cdot 10}$

$$\Leftrightarrow 2 = e^{R \cdot 10}$$

$$\ln(2) = R \cdot 10$$

$$R = \ln(2) / 10$$

$$\int_0^{20} V_0 e^{\frac{\ln(2)}{10} t} dt = V_0 \cdot \int_0^{20} e^{\frac{\ln(2)}{10} t} dt = V_0 \cdot e^{\frac{\ln(2)}{10} t} \cdot \frac{10}{\ln(2)} \Big|_{x=0}^{20}$$

$$= \left( \frac{40}{\ln(2)} - \frac{10}{\ln(2)} \right) \cdot V_0 = \frac{V_0 \cdot 30}{\ln(2)}$$

gemiddeld gebruik is dan:

$$\frac{V_0 \cdot 30}{\ln(2)} = \frac{V_0 \cdot 3}{2 \ln(2)}$$

$$(20 - 0)$$

Wanneer is gemiddeld verbruik gelijk aan verbruik?

$$\frac{V_0 \cdot 3}{2 \ln(2)} = V_0 \cdot e^{\frac{\ln(2)}{10} t}$$

$$\Leftrightarrow$$

$$\ln\left(\frac{3}{2 \ln(2)}\right) = \frac{\ln(2)}{10} \cdot t$$

$$t = \ln\left(\frac{3}{2 \ln(2)}\right) \cdot \frac{10}{\ln(2)} = 11,137 \text{ jaar}$$

$$6. \int 6000t^2 - 75t^4 = 2000t^3 - 15t^5 + C \quad (*)$$

Nu geldt voor  $t=0$  dat  $(*)$  gelijk moet zijn aan 1000

$$\rightarrow C = 1000$$

$$N(t) = 2000t^3 - 15t^5 + 1000$$

$$\begin{aligned} 7. \int_0^1 \frac{f(x)}{f(x) + f(1-x)} dx &= \int_0^1 \frac{f(x) + f(1-x)}{f(x) + f(1-x)} dx + \int_0^1 \frac{-f(1-x)}{f(x) + f(1-x)} dx \\ &= 1 + \int_0^1 \frac{f(y)}{f(1-y) + f(y)} dy \end{aligned}$$

$$\Leftrightarrow 2 \int_0^1 \frac{f(x)}{f(x) + f(1-x)} = 1$$

$$\Leftrightarrow \int_0^1 \frac{f(x)}{f(x) + f(1-x)} = 1/2$$

8.