

1. Review of Optimization Solution

1 Unconstrained optimization

Your beer producing company can advertise on both television (T) and radio (R). The effect of TV and radio commercials on sales is given by $S(T, R) = -T^2 + 40T - 2R^2 + 60R - 2TR + 20$. Determine the optimal number of T and R that maximize sales and calculate total sales at the optimum.

$$\max_{T, R} S(T, R) = -T^2 + 40T - 2R^2 + 60R - 2TR + 20$$

- **First order conditions**

Take the derivative of the objective function to the two variables to construct the FOCs:

$$\begin{aligned} & \begin{cases} \frac{\partial S(T, R)}{\partial T} = -2T + 40 - 2R = 0 \\ \frac{\partial S(T, R)}{\partial R} = -4R + 60 - 2T = 0 \end{cases} \\ \Leftrightarrow & \begin{cases} T = 20 - R \\ R = 15 - \frac{1}{2}T \end{cases} \end{aligned}$$

To solve the system of equations, insert the first equation into the second to solve for R :

$$\begin{aligned} R &= 15 - \frac{1}{2}(20 - R) \\ R - \frac{R}{2} &= 15 - 10 \\ R &= 10 \end{aligned}$$

Once you have found R , T is easily found as $T = 20 - R = 10$.

- **Second order conditions and test of the second derivative**

$$\frac{\partial^2 S(T, R)}{\partial T^2} = -2 < 0$$

and

$$\frac{\partial^2 S(T, R)}{\partial R^2} = -4 < 0$$

and

$$\frac{\partial^2 S(T, R)}{\partial T^2} \frac{\partial^2 S(T, R)}{\partial R^2} - \left(\frac{\partial^2 S(T, R)}{\partial T \partial R} \right)^2 = (-2)(-4) - (-2)^2 = 8 - 4 = 4 \geq 0.$$

Since the SOC are satisfied, $S(T, R)$ is concave and the solution from the FOC is a global maximum.

- **Total sales at optimal T and R**

$$S(10, 10) = -10^2 + 40 \cdot 10 - 2 \cdot 10^2 + 60 \cdot 10 - 2 \cdot 10 \cdot 10 + 20 = 520$$

2 Constrained optimization (one decision variable)

Your beer producing company can advertise only on television (T). The effect of TV commercials on sales is given by: $S(T) = -3T^2 + 42T + 200$. You have a budget of 400 euros and the price of one commercial is 40 euros. Determine the optimal level of TV commercials (T).

$$\begin{aligned} \max_T \quad & S(T) = -3T^2 + 42T + 200 \\ \text{s.t.} \quad & 40T - 400 \leq 0 \end{aligned}$$

Step 1: Assume that the constraint is binding, i.e. that it holds with equality $40T - 400 = 0$

$$\max_{T, \lambda} L(T, \lambda) = -3T^2 + 42T + 200 - \lambda(40T - 400)$$

FOCs:

$$\begin{cases} \frac{\partial L(T, \lambda)}{\partial T} = -6T + 42 - 40\lambda = 0 \\ \frac{\partial L(T, \lambda)}{\partial \lambda} = -40T + 400 = 0 \end{cases}$$

This leads to $T = 10$ and $\lambda = -\frac{9}{20}$

Step 2: Check if the assumption of binding constraint was correct, i.e. if $\lambda \geq 0$

For the constraint to be binding, λ must be greater than or equal to 0. But we found that $\lambda = -\frac{9}{20} < 0$, so the assumption of binding constraint was not correct! So we must set λ equal to zero and solve the problem again as an unconstrained maximization.

$$\max_T L(T) = -3T^2 + 42T + 200$$

FOC:

$$\frac{\partial L(T)}{\partial T} = -6T + 42 = 0$$

This leads to the solution $T = 7$

2. Market Equilibrium Solution

1 Own price elasticities

The demand and supply curves for coffee are given by $Q^d(P) = 600 - 2P$ and $Q^s(P) = 300 + 4P$.

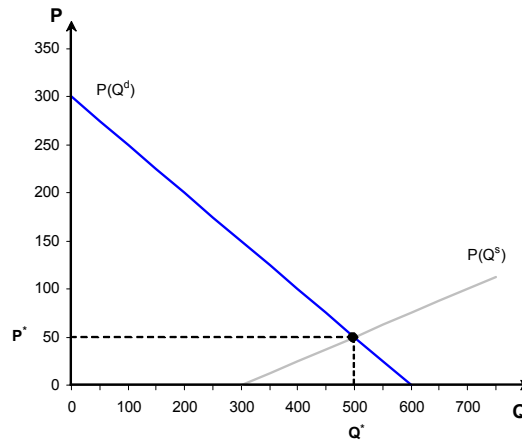
- a) Plot the supply and demand curves on a graph and show where the equilibrium occurs.
- b) Using algebra, determine the market equilibrium price and quantity of coffee.
- c) Calculate the own-price elasticity of demand and supply at the equilibrium.

a) Since we typically put P on the y-axis and Q on the x-axis, we rewrite

$$Q^d(P) = 600 - 2P \text{ as } P = \frac{Q^d - 600}{-2} = 300 - \frac{Q^d}{2}$$

$$Q^s(P) = 300 + 4P \text{ as } P = \frac{Q^s - 300}{4} = -75 + \frac{Q^s}{4}$$

Graphically:



b) In the equilibrium $Q^d = Q^s$,
 $600 - 2P^* = 300 + 4P^* \Rightarrow P^* = 50$

$$Q^{d*} = 600 - 2P^* = 600 - 2 \cdot 50 = 500$$

$$Q^{s*} = 300 + 4P^* = 300 + 4 \cdot 50 = 500$$

Therefore, the equilibrium in the form (Q^*, P^*) is $(500, 50)$.

c) The own-price elasticity of demand is $\varepsilon_{Q^d, P} = \frac{dQ^d}{dP} \cdot \frac{P}{Q^d}$.

From $Q^d(P)$ we find that $\frac{dQ^d}{dP} = -2$ such that $\varepsilon_{Q^d, P} = -2 \cdot \frac{50}{500} = -0.2$

The own-price elasticity of supply is $\varepsilon_{Q^s, P} = \frac{dQ^s}{dP} \cdot \frac{P}{Q^s}$.

From $Q^s(P)$ we find that $\frac{dQ^s}{dP} = 4$ such that $\varepsilon_{Q^s, P} = 4 \cdot \frac{50}{500} = 0.4$

3. Consumer Preferences

Solution

1 Problem 1

For the following sets of goods draw two indifference curves, U_1 and U_2 , with $U_2 > U_1$. Draw each graph placing the amount of the first good on the horizontal axis.

a) Hot dogs and chili (the consumer likes both and has a diminishing marginal rate of substitution of hot dogs for chili)

b) Sugar and Sweet'N Low (the consumer likes both and will accept an ounce of Sweet'N Low or an ounce of sugar with equal satisfaction)

c) Peanut butter and jelly (the consumer likes exactly 2 ounces of peanut butter for every ounce of jelly)

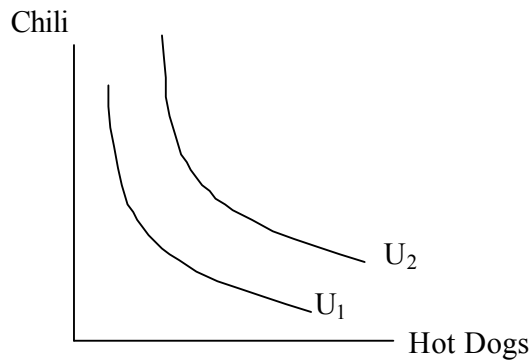
d) Nuts (which the consumer neither likes nor dislikes) and ice cream (which the consumer likes)

e) Apples (which the consumer likes) and liver (which the consumer dislikes)

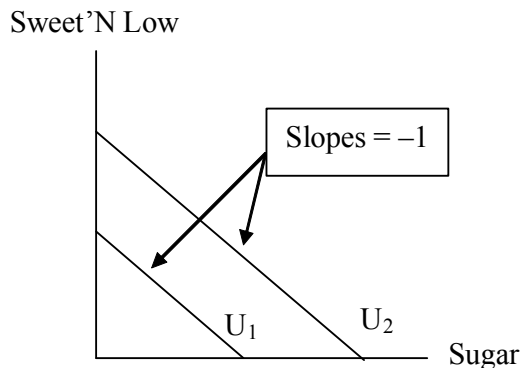
a) Likes both goods $\Rightarrow U_2$ lies above and to the right of U_1

Diminishing MRS:

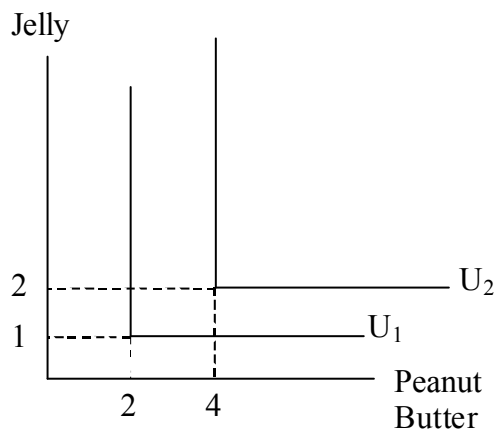
- With few hot dogs: willing to give up a lot of chili for 1 extra hot dog
- With many hotdogs: willing to give up a little bit of chili for 1 extra hot dog



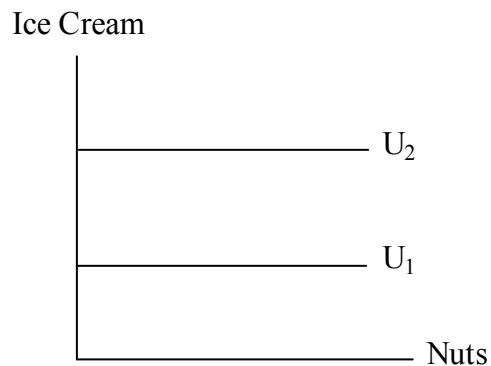
b) Likes both equally $\Rightarrow U_2$ lies above and to the right of U_1 & constant MRS (slope = -1) (perfect substitutes)



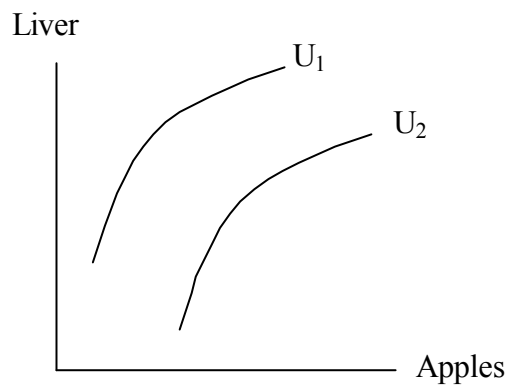
c) Constant proportions => perfect complements ($MRS=0$ or ∞)



d) Likes ice cream => more ice cream increases utility => U_2 lies above U_1
 Neutral to nuts => more nuts does not increase utility



e) Dislikes liver => more liver decreases utility => U_2 lies below U_1
 Likes apples => more apples increases utility => U_2 lies to the right of U_1
 => Violates Axiom 4: non-saturation



4. Consumer Choice Solution

1 Consumer choice

Helen buys 2 goods: clothing (x) and food (y). One unit of clothing costs 15 euros, while one unit of food costs 5 euros. Helen has a budget of 100 euros. Her preferences can be represented by the utility function $U(x, y) = \ln(xy)$. Determine the optimal bundle.

$$\begin{aligned} \max_{x, y} \quad & U(x, y) = \ln(xy) \\ \text{s.t.} \quad & 15x + 5y \leq 100 \end{aligned}$$

$$\text{Maximize Lagrangian: } \max_{x, y, \lambda} L(x, y, \lambda) = \ln(xy) - \lambda[15x + 5y - 100]$$

1. FOC assuming constraint is binding

$$\begin{aligned} (1) \quad & \frac{\partial L(x, y, \lambda)}{\partial x} = \frac{y}{xy} - 15\lambda = 0 \implies \frac{y}{xy} = 15\lambda \\ (2) \quad & \frac{\partial L(x, y, \lambda)}{\partial y} = \frac{x}{xy} - 5\lambda = 0 \implies \frac{x}{xy} = 5\lambda \\ (3) \quad & \frac{\partial L(x, y, \lambda)}{\partial \lambda} = -(15x + 5y - 100) = 0 \end{aligned}$$

$$\text{Dividing (1) by (2) we get } \frac{y/xy}{x/xy} = \frac{15\lambda}{5\lambda} \implies \frac{y}{x} = 3 \implies y = 3x$$

$$\text{Use } y = 3x \text{ in equation (3): } 15x + 5(3x) - 100 = 0 \implies x^* = \frac{10}{3} \implies y^* = 10$$

2. Check if assumption of binding constraint was correct, i.e. if $\lambda \geq 0$

$$\frac{y}{xy} - 15\lambda = 0 \implies \frac{10}{10(\frac{10}{3})} = 15\lambda \implies \frac{3}{15(10)} = \lambda > 0, \text{ therefore the constraint is binding.}$$

2 Consumer choice with quantity discount

Josh has a monthly income of 1500€. He spends this income on natural gas (G) and "other goods" (X). The first 1000 m^3 of gas always cost 0,4€ per m^3 . When buying more than 1000 m^3 , a reduction of 0,1€ per m^3 is given for the amount above 1000 m^3 . The unit price of the "other goods" is 0,25€. 1. Draw the budget set. 2. Determine the optimal bundle for Josh, given a utility function of $U(G, X) = GX$.

1. Draw the budget set

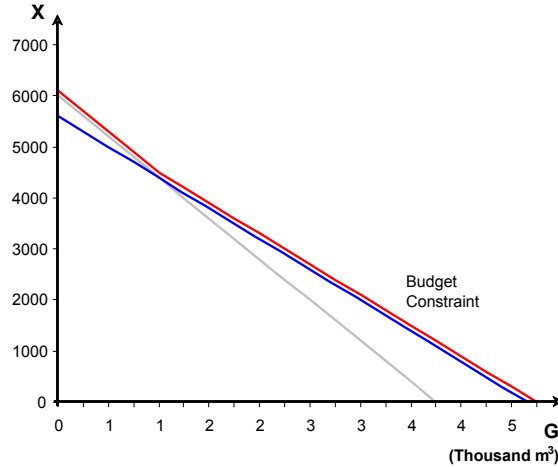
For $G \leq 1000$, Josh spends 0,4 G on gas.

$$\text{So for } G \leq 1000, \text{ the budget constraint is } 0,4G + 0,25X = 1500 \implies X = \frac{0,4G - 1500}{0,25} = 6000 - 1,6G$$

For $G > 1000$, Josh spends $(0,4)(1000) + (0,4 - 0,1)(G - 1000) = 100 + 0,3G$ on gas.

$$\text{So for } G > 1000, \text{ the budget constraint is } 100 + 0,3G + 0,25X = 1500 \implies X = \frac{1400 - 0,3G}{0,25} = 5600 - 1,2G$$

Graphically (see next page):



2. Determine optimal bundle

We don't know on which part of the budget constraint the solution will lie, so we have to solve the problem for both parts of the budget constraint.

a) Assuming Josh will buy less than 1000 m³ of gas

$$\max_{x,y} U(G, X) = GX$$

$$\text{s.t. } 0, 4G + 0, 25X - 1500 \leq 0$$

$$\text{Maximize Lagrangian: } \max_{X,G,\lambda} L = GX - \lambda [0, 4G + 0, 25X - 1500]$$

$$(1) \quad \frac{\partial L}{\partial X} = G - 0.25\lambda = 0 \implies \lambda = 4G$$

$$(2) \quad \frac{\partial L}{\partial G} = X - 0.4\lambda = 0 \implies \lambda = 2.5X$$

$$(3) \quad \frac{\partial L}{\partial \lambda} = -(0, 4G + 0, 25X - 1500) = 0$$

$$\text{Using (1) and (2) we get } 4G = 2.5X \implies G = \frac{5}{8}X$$

$$\text{Then using (3) we get } 0, 4\left(\frac{5}{8}X\right) + 0, 25X - 1500 = 0 \implies 0, 25X + 0, 25X = 1500 \implies X = 3000$$

This violates the assumption that Josh will buy less than 1000 m³ of gas. Therefore we now assume that Josh will buy more than 1000 m³ of gas and use the appropriate budget constraint.

b) Assuming Josh will buy more than 1000 m³ of gas

$$\max_{x,y} U(G, X) = GX$$

$$\text{s.t. } 0, 3G + 0, 25X - 1400 \leq 0$$

$$\text{Maximize Lagrangian: } \max_{X,G,\lambda} L = XG - \lambda [0, 3G + 0, 25X - 1400]$$

$$(1) \quad \frac{\partial L}{\partial X} = G - 0.25\lambda = 0 \implies \lambda = 4G$$

$$(2) \quad \frac{\partial L}{\partial G} = X - 0.3\lambda = 0 \implies \lambda = \frac{10}{3}X$$

$$(3) \quad \frac{\partial L}{\partial \lambda} = -(0, 3G + 0, 25X - 1400) = 0$$

$$\text{Using (1) and (2) we get } 4G = \frac{10}{3}X \implies G = \frac{5}{6}X$$

$$\text{Then using (3) we get } 0, 3\left(\frac{5}{6}X\right) + 0, 25X - 1400 = 0 \implies 0, 25X + 0, 25X = 1400 \implies X = 2800$$

This confirms the assumption that Josh will buy more than 1000 m³ of gas.

$$\text{Now } G = \frac{5}{6}2800 = \frac{7000}{3}$$

Finally we need to check $\lambda = 4G > 0$, therefore the constraint is indeed binding.

3 Intertemporal consumer choice

Jack makes his consumption and saving decisions for the next two months. His income this month is \$1000, and he knows that he will get a raise next month making his income \$1050. The current interest rate (at which he is free to borrow or lend) is 5%. Denoting this month's consumption by x and next month's by y , Jack's utility function is $U(x, y) = xy^2$. How much will Jack consume this month and next month?

$$\begin{aligned} \max_{x,y} \quad & U(x, y) = xy^2 \\ \text{s.t.} \quad & x + \frac{y}{1,05} - (1000 + \frac{1050}{1,05}) \leq 0 \end{aligned}$$

$$\text{Maximize Lagrangian: } \max_{x,y,\lambda} L = xy^2 - \lambda \left[x + \frac{y}{1,05} - 2000 \right]$$

1. FOC assuming constraint is binding

$$\begin{aligned} (1) \quad & \frac{\partial L}{\partial x} = y^2 - \lambda = 0 \implies y^2 = \lambda \\ (2) \quad & \frac{\partial L}{\partial y} = 2xy - \frac{\lambda}{1,05} = 0 \implies 2xy = \frac{\lambda}{1,05} \\ (3) \quad & \frac{\partial L}{\partial \lambda} = -(x + \frac{y}{1,05} - 2000) = 0 \end{aligned}$$

$$\text{Dividing (1) by (2) we get } \frac{y^2}{2xy} = \frac{\lambda}{\frac{\lambda}{1,05}} \implies \frac{y^2}{2xy} = 1,05 \implies y = (2,1)x$$

$$\text{Use } y = (2,1)x \text{ in equation (3): } x + 2x = 2000 \implies x^* = \frac{2000}{3} \implies y^* = 1400$$

2. Check if assumption of binding constraint was correct, i.e. if $\lambda \geq 0$

$$\lambda = y^2 \implies \lambda = (1400)^2 > 0, \text{ therefore the constraint is binding.}$$

Since $x = \frac{2000}{3} < I_1 = 1000$ and $y = 1400 > I_2 = 1050$, Jack saves money in period 1 for later consumption in period 2.

5. Demand Theory

Solution

1 Derivation of demand for goods

Consider a consumer with the following utility function: $U(x, y) = 10x - x^2 + y$. Suppose furthermore that the income of the consumer equals $I = 20$ and let $P_y = 1$.

- Derive the demand for goods x and y from the utility maximization problem
- Compute the consumer surplus at $P_x = 5$.
- Calculate the change in the consumer surplus when the price P_x drops from 5 to 3.

a) Derive the demand

$$\begin{aligned} \max_{x,y} U(x, y) &= 10x - x^2 + y \\ \text{s.t. } P_x x + y - 20 &\leq 0 \end{aligned}$$

Maximize Lagrangian: $\max_{x,y,\lambda} L(x, y, \lambda) = 10x - x^2 + y - \lambda(P_x x + y - 20)$

$$\begin{aligned} (1) \quad \frac{\partial L}{\partial x} &= 10 - 2x - \lambda P_x = 0 \\ (2) \quad \frac{\partial L}{\partial y} &= 1 - \lambda = 0 \implies \lambda = 1 \\ (3) \quad \frac{\partial L}{\partial \lambda} &= -P_x x - y + 20 = 0 \end{aligned}$$

Using (1) and (2): $10 - 2x - P_x = 0 \implies P_x = 10 - 2x$. This is the inverse demand function. The regular demand function is then $x = 5 - \frac{P_x}{2}$.

$$\text{Using this and (3) } P_x \left(5 - \frac{P_x}{2}\right) + y - 20 = 0 \implies 5P_x - \frac{P_x^2}{2} + y = 20 \implies y = 20 - 5P_x + \frac{P_x^2}{2}$$

Finally, we check $\lambda = 1 > 0$ so the constraint is indeed binding.

b) Compute consumer surplus

- Using the inverse demand function ($P_x = 10 - 2x$):

Consumer surplus is maximum WTP minus actual expenditures, or the area under the inverse demand function minus actual expenditures. The area under the inverse demand curve is $T(x) = \int_0^x (10 - 2u) du = 10x - x^2 = (10 - x)x$. So consumer surplus is $CS(x) = T(x) - P_x x = (10 - x)x - (10 - 2x)x = x^2$. At $P_x = 5$, we have $x = 5 - \frac{5}{2} = \frac{5}{2}$, so $CS(x) = x^2 = \frac{25}{4}$.

- Using the regular demand function ($x = 5 - \frac{P_x}{2}$):

Consumer surplus can also be computed as the area under the regular demand function. It is then a function of the price rather than the quantity consumed. The maximum price (at which demand becomes zero) is $P_x = 10 - 2 \times 0 = 10$. So $CS(P_x) = \int_{P_x}^{10} \left(5 - \frac{u}{2}\right) du = \left(5u - \frac{1}{4}u^2\right) \Big|_{P_x}^{10} = \left(50 - \frac{1}{4}100\right) - \left(5P_x - \frac{1}{4}P_x^2\right) = 25 - 5P_x + \frac{1}{4}P_x^2$. At $P_x = 5$ the consumer surplus is $CS = 25 - 5(5) + \frac{5^2}{4} = \frac{25}{4}$.

c) Calculate change in consumer surplus

If $P_x = 3$ then the consumer surplus is: $CS = 25 - 5(3) + \frac{3^2}{4} = \frac{49}{4}$ so that the change in consumer surplus when P_x drops from 5 to 3 is an increase by 6 (i.e. $\frac{49}{4} - \frac{25}{4} = \frac{24}{4} = 6$)

2 Derivation of demand for labour

Consider the following utility function for Joris: $U(C, Z) = 48Z + ZC - Z^2$, with Z = quantity of leisure, L = quantity of labour, T = available time units, C = consumption, P = price of 1 unit of the compound good, W = nominal wage and A = non-labour income.

- Determine the demand function for leisure (from utility maximization) and the labour supply function.
- Show that $\frac{dL}{dw}$ does not always have the same sign and deduce the sign conditions.
- Calculate the number of hours of work given the following parameter values: $T = 24$ hours – 10 hours sleep = 14 hours, $P = 1$ euro, $w = 10$ euro per working hour, $A = 5$ euro per day.
- What are your conclusions in terms of the shape of the labour supply curve for the specific case of c)?

$$\begin{aligned} \text{a)} \max_{C, Z} U(C, Z) &= 48Z + ZC - Z^2 \\ \text{s.t. } PC - wL - A &\leq 0 \text{ and } L = T - Z \end{aligned}$$

$$\text{Maximize Lagrangian } \max_{c, z, \lambda} L(C, Z, \lambda) = 48Z + ZC - Z^2 - \lambda(PC - w(T - Z) - A)$$

$$\begin{aligned} (1) \quad \frac{\partial L}{\partial Z} &= 48 + C - 2Z - w\lambda = 0 \\ (2) \quad \frac{\partial L}{\partial C} &= Z - P\lambda = 0 \implies \lambda = \frac{Z}{P} \\ (3) \quad \frac{\partial L}{\partial \lambda} &= -PC + w(T - Z) + A = 0 \end{aligned}$$

$$\text{Using (1) and (2): } 48 + C - 2Z - w\left(\frac{Z}{P}\right) = 0 \implies C = \frac{w}{P}Z + 2Z - 48.$$

$$\text{Using this result in (3): } Z^* = \frac{48P + wT + A}{2w + 2P} \text{ (we don't need to calculate } C^* \text{).}$$

$$\text{The labour supply curve is given by } L^* = T - Z^* = T - \frac{48P + wT + A}{2w + 2P}.$$

$$\text{Finally, we check } \lambda = \frac{Z^*}{P} = \frac{48P + wT + A}{2wP + 2P^2} > 0.$$

$$\begin{aligned} \text{b)} \quad L^* &= T - \frac{48P + wT + A}{2w + 2P} \text{ (note: derivative of a quotient } \frac{f}{g} = \frac{f'g - fg'}{g^2}) \\ \frac{\partial L^*}{\partial w} &= -\frac{2(w+P)T - (48P + wT + A)2}{4(P+w)^2} = -\frac{2wT + 2PT - 96P - 2wT - 2A}{4(P+w)^2} = -\frac{2PT - 96P - 2A}{4(P+w)^2} = \frac{48P - TP + A}{2(P+w)^2} \end{aligned}$$

- $\frac{\partial L^*}{\partial w} = 0$ when $48P - TP + A = 0$. Labour supply is not sensitive to wages.
- $\frac{\partial L^*}{\partial w} > 0$ when $48P - TP + A > 0$. Labour supply increases when wage increases.
- $\frac{\partial L^*}{\partial w} < 0$ when $48P - TP + A < 0$. Labour supply decreases when wage increases = backward bending labour supply.

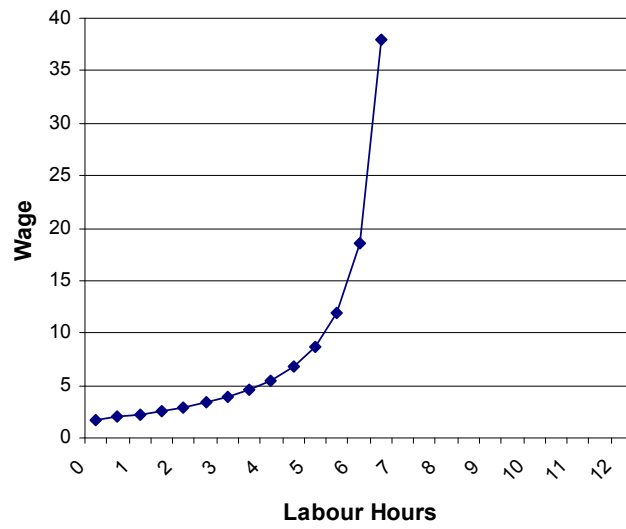
$$\text{c)} \text{ Given } T = 14, P = 1, A = 5 \text{ and } w = 10 \text{ we get } L^* = 14 - \frac{48 + 10(14) + 5}{2(10) + 2} = 5,23, \text{ i.e. } 5\text{h}14\text{min.}$$

$$\text{d)} \quad \frac{\partial L^*}{\partial w} = \frac{48P - TP + A}{2(P+w)^2} = \frac{48 - 14 + 5}{2(1+10)^2} > 0, \text{ so the labour supply function is increasing in wages.}$$

Some points to draw:

- $w = 0 \implies L^* = 14 - \frac{48 + 5}{2} = \frac{28 - 53}{2} < 0 \implies L^* = 0$
- $w = 5 \implies L^* = 14 - \frac{48 + (14 \cdot 5) + 5}{2(5+1)} = \frac{168 - 123}{12} = \frac{45}{12} \implies L^* = 3.75$
- $w = 10 \implies L^* = 14 - \frac{48 + (14 \cdot 10) + 5}{2(10+1)} = \frac{308 - 193}{22} = \frac{115}{22} \implies L^* = 5.23$

Labour Supply Curve



6. Inputs and production functions

Solution

1 Linear production function

Take the following production function (with a and b positive): $Q(L, K) = aL + bK$

a) What is the marginal productivity of labour and of capital?

b) What is the marginal rate of technical substitution? Interpret in terms of the substitution elasticity.

a)

$$\begin{aligned}MP_L(L, K) &= \frac{\partial Q(L, K)}{\partial L} = a \\MP_K(L, K) &= \frac{\partial Q(L, K)}{\partial K} = b\end{aligned}$$

The marginal productivity of both inputs is constant. Employing 1 extra unit of an input L (resp. K) will always increase the output by the same amount a (resp. b), irrespective of the amount of inputs that are already used in the production process.

b)

$$MRTS_{L,K}(L, K) = \frac{MP_L(L, K)}{MP_K(L, K)} = \frac{a}{b}$$

The MTRS is a constant. We can trade in $\frac{a}{b}$ units of capital for 1 unit of labour to maintain the same output level. Since this is a constant, we can conclude that L and K are perfect substitutes. We know that perfect substitutes have a substitution elasticity of infinity, since no matter how much labour is already used, the 'trade rate' with capital stays the same.

$$\sigma = \frac{\text{percentage change in } K/L}{\text{percentage change in } MRTS_{L,K}(L, K)} = \frac{\cdot}{0} = \infty$$

7. Cost Functions

Solution

1 Production function with two inputs

Consider the following production function $Q = \sqrt{L} + \sqrt{K}$

- Derive the conditional demand of both inputs.
- Check Shepards Lemma for the total cost function
- Check if $\lambda = MC$. What does this imply?
- Is the total cost function characterized by (dis)economies of scale? What does this imply?
- Calculate the scale elasticity of production and interpret.
- Calculate the cost elasticity of the output. What is the relation to the scale elasticity?

$$\text{a) } \min_{L,K} TC = w.L + r.K$$

$$\text{s.t. } \sqrt{L} + \sqrt{K} \geq Q$$

$$\text{Minimize Lagrangian: } \min_{L,K,\lambda} \mathcal{L} = w.L + r.K - \lambda.(\sqrt{L} + \sqrt{K} - Q)$$

$$(1) \quad \frac{\partial \mathcal{L}}{\partial L} = w - \frac{\lambda}{2\sqrt{L}} = 0 \implies \lambda = 2w\sqrt{L}$$

$$(2) \quad \frac{\partial \mathcal{L}}{\partial K} = r - \frac{\lambda}{2\sqrt{K}} = 0 \implies \lambda = 2r\sqrt{K}$$

$$(3) \quad \frac{\partial \mathcal{L}}{\partial \lambda} = \sqrt{L} + \sqrt{K} - Q = 0$$

Using (1) and (2), we get $2w\sqrt{L} = 2r\sqrt{K} \implies \sqrt{L} = \frac{r}{w}\sqrt{K}$

Substituting this in (3), we get $Q = \sqrt{L} + \sqrt{K} = \frac{r}{w}\sqrt{K} + \sqrt{K} = \sqrt{K} \cdot \left(\frac{r}{w} + 1\right)$

$$\implies K^*(Q, w, r) = \left(\frac{Q}{\left(\frac{r}{w} + 1\right)}\right)^2$$

$$\implies L^*(Q, w, r) = \left(\frac{Q}{\left(\frac{w}{r} + 1\right)}\right)^2$$

Check $\lambda = 2w\sqrt{L} = 2w \frac{Q}{\left(\frac{w}{r} + 1\right)} = \frac{2wrQ}{w+r} > 0$, so the constraint was indeed binding.

b) The total cost function is given by substituting the conditional demands in $TC = w.L^* + r.K^*$

$$\begin{aligned} TC(Q, w, r) &= w \cdot \left(\frac{Q}{\left(\frac{w}{r} + 1\right)}\right)^2 + r \cdot \left(\frac{Q}{\left(\frac{r}{w} + 1\right)}\right)^2 \\ &= w \left(\frac{rQ}{w+r}\right)^2 + r \left(\frac{wQ}{r+w}\right)^2 \\ &= \frac{wrQ^2}{(w+r)^2} \cdot (r+w) \\ &= \frac{wrQ^2}{(w+r)} \end{aligned}$$

Shepard's Lemma states $\frac{\partial TC(Q, w, r)}{\partial w} = L^*(Q, w, r)$ (and similar for capital).

$$\begin{aligned}
 \frac{\partial TC(Q, w, r)}{\partial w} &= \frac{(w+r)rQ^2 - 1 \cdot wrQ^2}{(w+r)^2} \\
 &= \frac{rQ^2(w+r-w)}{(w+r)^2} \\
 &= \frac{r^2Q^2}{(w+r)^2} \\
 &= \left(\frac{Q}{\left(\frac{w}{r} + 1\right)} \right)^2 = L^*(Q, w, r)
 \end{aligned}$$

c) From a) we know that $\lambda = \frac{2wrQ}{w+r}$. From the total cost function, we find $MC(Q, w, r) = \frac{\partial TC(Q, w, r)}{\partial Q} = \frac{2wrQ}{(w+r)}$. We indeed find that the Lagrange multiplier equals the above marginal cost function!

The Lagrange multiplier gives the change in the value of the objective function when the constraint changes slightly. In terms of this optimization problem, an increase in the output level (constraint) will result in an increase in the total costs (objective function) with a value λ . It is therefore intuitive that this value equals the marginal costs.

d) There are diseconomies of scale:

$$\begin{aligned}
 AC(Q, w, r) &= \frac{TC(Q, w, r)}{Q} = \frac{wrQ}{(w+r)} \\
 \frac{\partial AC(Q, w, r)}{\partial Q} &= \frac{wr}{(w+r)} > 0
 \end{aligned}$$

e) There are decreasing returns to scale:

$$\begin{aligned}
 \varepsilon_{Q, \text{Len}K}(L, K) &= MP_L \cdot \frac{L}{Q(L, K)} + MP_K \cdot \frac{K}{Q(L, K)} \\
 &= \frac{1}{2\sqrt{L}} \cdot \frac{L}{\sqrt{L} + \sqrt{K}} + \frac{1}{2\sqrt{K}} \cdot \frac{K}{\sqrt{L} + \sqrt{K}} \\
 &= \frac{1}{2(\sqrt{L} + \sqrt{K})} (\sqrt{L} + \sqrt{K}) = \frac{1}{2} < 1
 \end{aligned}$$

For all values of K and L, an increase of 1% in both K and L only leads to an increase in output of 0.5%.

f) There are decreasing returns to scale:

$$\begin{aligned}
 \varepsilon_{TC, Q}(Q, w, r) &= \frac{\partial TC(Q, w, r)}{\partial Q} \frac{Q}{TC(Q, w, r)} \\
 &= \frac{2wrQ}{(w+r)} \cdot \frac{Q}{\frac{wrQ^2}{(w+r)}} \\
 &= \frac{2wrQ^2}{wrQ^2} = 2 > 1
 \end{aligned}$$

For all values of K and L, increasing the output by 1% results in an increase in total costs of 2%. The cost elasticity $\varepsilon_{TC, Q}(Q) = 2$ is the inverse of the scale elasticity $\varepsilon_{Q, \text{Len}K}(L, K) = 0.5$.

8. Cost Curves Solution

1 Short run and the relation with the long run

Consider the following production function: $Q = \sqrt{L}\sqrt{K}$. Suppose that in the short run $K = \bar{K} = 4$ is fixed, so that L is the only variable input.

- a) Discuss the relation between Q and L and draw Q as a function of L .
- b) Derive the short run conditional demand function for L
- c) Derive the short run total cost function. Does this function exhibit (dis)economies of scale?
- d) Assume $w = 1$ and $r = 1$. Derive the long run total cost function from the cost minimization problem.

Draw the short run and long run cost function. When are both costs equal?

a) Given $K = \bar{K} = 4$, $Q = \sqrt{L}\sqrt{K} = 2\sqrt{L}$

- $\frac{\partial Q(L)}{\partial L} = 2 \cdot \frac{1}{2} L^{-1/2} = \frac{1}{\sqrt{L}} > 0$: the production function is increasing in the input L
- $\frac{\partial^2 Q(L)}{\partial L^2} = 1 \cdot \frac{-1}{2} L^{-3/2} = \frac{-1}{2\sqrt{L}^3} < 0$: the production function is concave

b) $\min_L TC = wL + r\bar{K} = wL + r4$
s.t. $Q = 2\sqrt{L}$

With only 1 decision variable, the labour demand follows directly solving the constraint: $L^*(Q) = \left(\frac{Q}{2}\right)^2$. (Since there are no inputs to substitute, the conditional demand does not depend on the input prices.)

- c) Substitute the demand for labor in the short run cost function:

$$STC\left(Q, \bar{K}, w, r\right) = wL^*\left(Q, \bar{K}\right) + r\bar{K} = w\left(\frac{Q}{2}\right)^2 + 4r$$

For the economies of scale, we look at the derivative of the short run average cost curve:

- Step 1. the average cost curve: $SAC(Q, w, r) = \frac{STC(Q, w, r)}{Q} = w\frac{Q}{4} + \frac{4r}{Q}$
- Step 2. the derivative of the average cost curve: $\frac{\partial SAV(Q, w, r)}{\partial Q} = \frac{w}{4} - \frac{4r}{Q^2}$
 $\frac{\partial SAV(Q, w, r)}{\partial Q} > 0 \iff \frac{w}{4} - \frac{4r}{Q^2} > 0 \iff Q > \sqrt{\frac{16r}{w}}$

This function exhibits diseconomies of scale when $Q > \sqrt{\frac{16r}{w}}$.

d) First solve for the long run demand functions, then derive the long run total cost function

1. demand functions

$$\min_{L,K} TC = 1.L + 1.K$$

$$s.t. Q = L^{1/2}K^{1/2}$$

$$\min_{L,K,\lambda} \mathcal{L} = L + K - \lambda(L^{1/2}K^{1/2} - Q)$$

$$(1) \quad \frac{\partial \mathcal{L}}{\partial L} = 1 - \frac{\lambda K^{1/2}}{2L^{1/2}} = 0 \implies \lambda = \frac{2L^{1/2}}{K^{1/2}}$$

$$(2) \quad \frac{\partial \mathcal{L}}{\partial K} = 1 - \frac{\lambda L^{1/2}}{2K^{1/2}} = 0 \implies \lambda = \frac{2K^{1/2}}{L^{1/2}}$$

$$(3) \quad \frac{\partial \mathcal{L}}{\partial \lambda} = L^{1/2}K^{1/2} - Q = 0$$

$$\text{From (1) and (2), we get } \frac{2L^{1/2}}{K^{1/2}} = \frac{2K^{1/2}}{L^{1/2}} \implies L = K$$

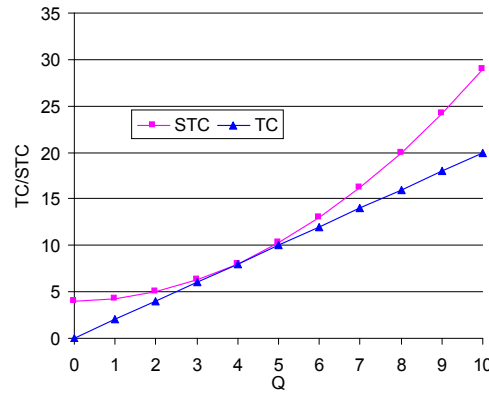
$$\text{Using this result in (3), we get } L^{1/2}L^{1/2} = Q \implies L^* = Q \text{ and } K^* = Q$$

2. Derive long run total cost function

$$TC(L^*, K^*, w = 1, r = 1) = w.L^* + r.K^* = Q + Q = 2Q$$

3. Graph of short run and long run cost function

- $TC(L^*, K^*, w = 1, r = 1) = 2Q$: increasing in Q and straight.
- $STC\left(Q, \bar{K}, w = 1, r = 1\right) = w\left(\frac{Q}{2}\right)^2 + 4r = \frac{Q^2}{4} + 4$: increasing in Q and convex ($\frac{\partial^2 STC}{\partial Q^2} = \frac{1}{2}$).
- The costs are the same when $2Q = \frac{Q^2}{4} + 4 \iff \frac{Q^2}{4} - 2Q + 4 = 0$. Solve this quadratic equation (remember: $aX^2 + bX + c = 0 \rightarrow X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$) to $Q = 4$. When $Q = 4$, the optimal choice of capital in the long run (K^*) equals the short run amount of capital ($\bar{K} = 4$).



9&10. Perfect Competition and Government intervention

Solution

1 Excise tax on consumer

The market demand for DVD-players is given by $D(P) = 300 - P$ and the market supply is characterized by $S(P) = 0.5P - 30$. In both expressions, P is the price per unit of a DVD-player. The government introduces an excise tax (recupelbijdrage) of 6€ per DVD-player, that is added to the bill of the consumer.

- a) What is the equilibrium price and quantity before the government intervenes in the market?
- b) What is the equilibrium price and quantity after the government intervenes in the market?
- c) Calculate changes in consumer surplus, producer surplus and welfare.
- d) Illustrate graphically and recalculate CS, PS and welfare using the graph.

$$a) D(P) = S(P) \iff 300 - P^* = 0.5P^* - 30 \Rightarrow P^* = 220 \Rightarrow Q^* = 80$$

- b) (1) Demand $D(P_C) = 300 - P_C$
- (2) Supply $S(P_P) = 0.5P_P - 30$
- (3) $P_C = P_P + 6$

Solve this system of equations by substituting (3) in (1) and equating this to (2):

$$300 - (P_P^* + 6) = 0.5P_P^* - 30 \implies 300 + 30 - 6 = 1.5P_P^* \implies P_P^* = 216 \implies P_C^* = 222 \implies Q^* = 78$$

c) **Step 1. CS and PS before the excise tax** ($Q^* = 80$)

- CS is total WTP (area under inverse demand function $P(Q) = 300 - Q$) minus expenditures (PQ):

$$CS = T(Q) - P(Q)Q = \int_0^Q P(u)du - P(Q)Q = \int_0^Q (300 - u)du - (300 - Q)Q = \left(300u - \frac{1}{2}u^2\right)\Big|_0^Q - (300 - Q)Q = 300Q - \frac{1}{2}Q^2 - (300 - Q)Q = \frac{Q^2}{2} = \frac{80^2}{2} = 3200$$

- PS is revenues (PQ) minus costs (area under inverse supply function ($P(Q) = 60 + 2Q$):

$$PS = P(Q)Q - \int_0^Q P(u)du = (60 + 2Q)Q - \int_0^Q (60 + 2u)du = (60 + 2Q)Q - (60u + u^2)\Big|_0^Q = (60 + 2Q)Q - (60Q + Q^2) = Q^2 = 80^2 = 6400$$

- Government revenue: $G = 0$

- Welfare: $W = CS + PS + G = 9600$

Step 2. CS and PS after the excise tax ($Q^* = 78$)

$$- CS = \frac{1}{2}Q^2 = \frac{1}{2}(78^2) = 3042$$

$$- P(Q) = Q^2 = 6084$$

$$- G = Q * 6 = 78 * 6 = 468$$

$$- W = CS + PS + G = 9594$$

Step 3. Change in CS, PS and welfare

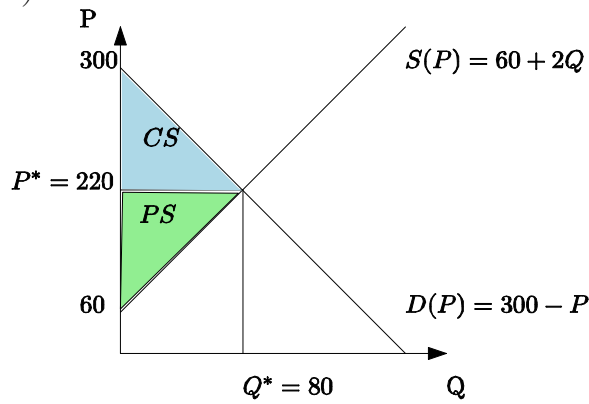
$$- \Delta CS = 3042 - 3200 = -158 < 0$$

$$- \Delta PS = 6084 - 6400 = -316 < 0$$

$$- \Delta G = 468 > 0$$

$$- \Delta W = -158 - 316 + 468 = 9594 - 9600 = -6$$

d)



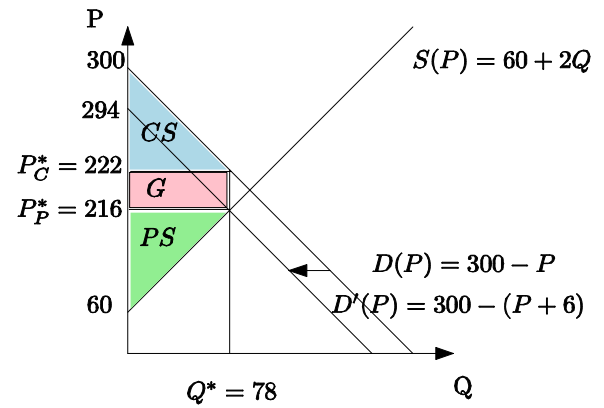
Before the excise tax (left panel)

$$CS = (300 - 220) \times 80 / 2 = 3200$$

$$PS = (220 - 60) \times 80 / 2 = 6400$$

$$G = 0$$

$$W = CS + PS + G = 9600$$



After the excise tax (right panel)

$$CS = (300 - 222) \times 78 / 2 = 3042$$

$$PS = (216 - 60) \times 78 / 2 = 6084$$

$$G = (222 - 216) \times 78 = 468$$

$$W = CS + PS + G = 9594$$

11. Monopoly Solution

1 Problem 1

A monopolist faces an inverse demand of $P = a - bQ$ and total costs of $TC(Q) = f + cQ + \frac{d}{2}Q^2$. Assume $a, b, c > 0$ and $a > c$, but d can be positive or negative ($MC(Q)$ can be increasing or decreasing).

- What is the optimal quantity and price in terms of a, b, c and d ?
- Give a sufficient condition that makes sure that your solution is a maximum. Interpret.
- Interpret the effects of a and c on the equilibrium quantity and price.
- Compare these results with perfect competition. Assume now that $a, b, c, d > 0$.

a) For the monopolist: $\frac{d\pi(Q)}{dQ} = 0 \Leftrightarrow MR(Q) = MC(Q)$

$$MR(Q) = \frac{d}{dQ}(TR(Q)) = \frac{d}{dQ}(P(Q) \cdot Q) = \frac{d}{dQ}(aQ - bQ^2) = a - 2bQ$$

$$MC(Q) = \frac{d}{dQ}(TC(Q)) = \frac{d}{dQ}(f + cQ + \frac{d}{2}Q^2) = c + dQ$$

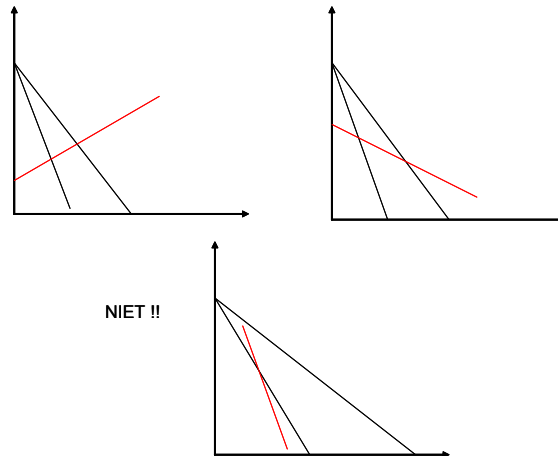
$$\Rightarrow Q^* = \frac{a-c}{d+2b} \Rightarrow P^* = \frac{ad+ab+bc}{d+2b}$$

b) The second derivative of the profit function must be negative:

$$\frac{d^2\pi(Q)}{dQ^2} = \frac{d^2TR(Q)}{dQ^2} - \frac{d^2TC(Q)}{dQ^2} < 0 \Leftrightarrow \frac{dMR(Q)}{dQ} < \frac{dMC(Q)}{dQ} \Leftrightarrow -2b < d$$

This condition states that the marginal cost function should intersect with the marginal revenue function from below, i.e. marginal cost should not be too decreasing in output! Graphically we can look at two cases:

- If $d > 0$, then MC increase with output and the condition will be automatically satisfied (since $b > 0$).
- If $d < 0$, then MC decrease with output. The condition then says that the MC curve has to be less steep than the MR curve (intersect from below).



c) An increase in c is an upward shift of marginal costs. An increase in a is an upward shift of demand. Remember that $d + 2b > 0$ (from the second order condition).

- $\frac{dQ^*}{da} = \frac{1}{d+2b} > 0$: output increases if demand increases
- $\frac{dQ^*}{dc} = \frac{-1}{d+2b} < 0$: output decreases if marginal costs increase
- $\frac{dP^*}{dc} = \frac{b}{d+2b} > 0$: price increases if marginal costs increase
- $\frac{dP^*}{da} = \frac{d+b}{d+2b} \geq 0 \Leftrightarrow d+b \geq 0$
 $d+b > 0 \Leftrightarrow d > 0$ (MC is increasing) or $d > -b$ (MC not too decreasing)
 $d+b < 0 \Leftrightarrow d < -b$ (MC is sufficiently downward sloping, but it cannot be too downward sloping since the second-order condition should be satisfied, so we need $-2b < d < -b$).

\Rightarrow A demand increase leads to higher output (since $\frac{dQ^*}{da} > 0$), which leads to a price increase, unless the marginal cost drops sufficiently (d sufficiently negative).

d) With perfect competition, $P(Q) = MC(Q) \Leftrightarrow a - bQ = c + dQ \Rightarrow Q^* = \frac{a-c}{d+b} \Rightarrow P^* = \frac{ad+bc}{d+b}$

- $\frac{dQ^*}{dc} = \frac{-1}{d+b} < 0$ & $\frac{dQ^*}{da} = \frac{1}{d+b} > 0$: The cost and demand effects are larger under perfect competition than under monopoly, since the denominator is smaller.
- $\frac{dP^*}{dc} = \frac{b}{d+b} > 0$: The effect of a cost increase c on price is stronger under perfect competition than under monopoly, since the denominator is smaller.
- $\frac{dP^*}{da} = \frac{d}{d+b} > 0$: The effect of a demand increase on price would be stronger under competition than under monopoly if $\frac{d}{d+b} > \frac{d+b}{d+2b} \Leftrightarrow d^2 + 2bd > d^2 + b^2 + 2bd \Leftrightarrow 0 > b^2$. This is not the case, so the effect of a demand increase on price is stronger under *monopoly* than under competition.

2 Problem 2

A monopolist faces demand of $Q(P) = 360 - 20P$ and total costs of $TC(Q) = 6Q + 0.05Q^2$

a) Calculate optimal price and quantity of the monopolist. Check the three optimality conditions and give a graphical representation.

b) Calculate price elasticity of demand and the Lerner index in equilibrium

c) Repeat question a) under perfect competition, assuming that the marginal cost curve of the monopolist coincides with the supply function under perfect competition.

d) Evaluate the transition from perfect competition to monopoly from a welfare point of view. + Graph.

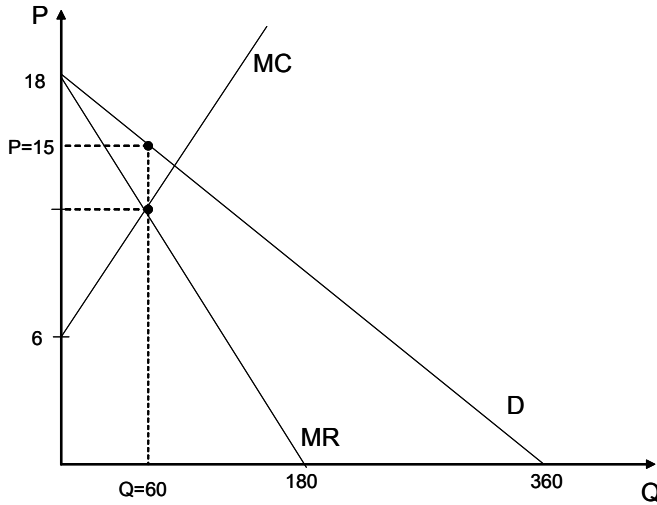
a) $\max_Q \pi(Q) = P(Q)Q - TC(Q)$

To get $P(Q)$, invert $Q(P)$: $P = 18 - 0.05Q \Rightarrow \pi(Q) = (18 - 0.05Q)Q - (6Q + 0.05Q^2) = 12Q - 0.1Q^2$

$\frac{d\pi(Q)}{dQ} = 12 - 0.2Q = 0 \Rightarrow Q^* = 60 \Rightarrow P^* = 15$

Three optimality conditions:

1. FOC: $\frac{d\pi(Q^*)}{dQ} = 0$, this is indeed the case (see a)
2. SOC $\frac{d^2\pi(Q^*)}{dQ^2} < 0$, this is indeed the case since $\frac{d^2\pi(Q^*)}{dQ^2} = -0.2$
3. $P > AC(Q)$, this is indeed the case since $AC(Q) = \frac{TC(Q)}{Q} = 6 + 0.05Q = 6 + 0.05 \times 60 = 9 < P = 15$



b) **Price elasticity of demand:** $\varepsilon_{Q^d, P} = \frac{dQ^d(P)}{dP} \frac{P^*}{Q^*} = -20 * \frac{15}{60} = -5$. Demand is elastic, which must be the case under monopoly when marginal costs are positive.

Lerner index: $L = \frac{P(Q^*) - MC(Q^*)}{P(Q^*)} = \frac{15 - (6 + 0.1 \times 60)}{15} = 0.2$. For a monopolist, the Lerner index is the inverse of the price elasticity of demand.

c) $\max_Q \pi(Q) = TR(Q) - TC(Q) = PQ - TC(Q)$ (Now every producer takes the price as given!)

$$\frac{d\pi(Q)}{dQ} = 0 \Leftrightarrow P = MC(Q)$$

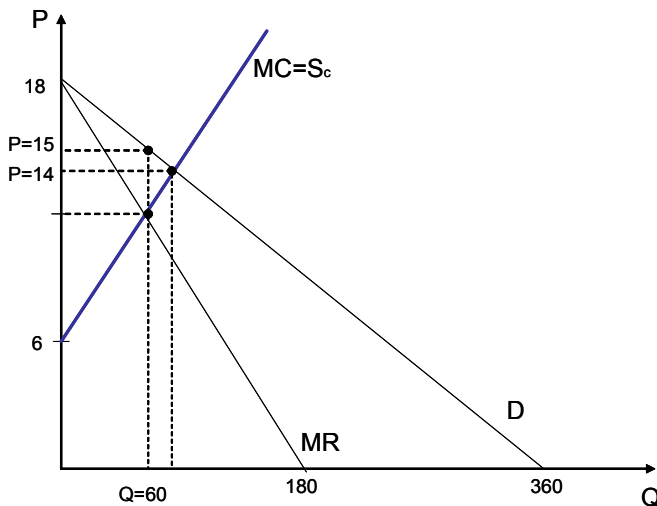
$$\Leftrightarrow 18 - 0.05Q^o = 6 + 0.1Q^o$$

$$\Rightarrow Q = 80 \text{ (This is now total quantity produced by all firms in the market.)}$$

$$\Rightarrow P = 14$$

Three optimality conditions:

1. FOC: $\frac{d\pi(Q^*)}{dQ} = 0$, this is indeed the case (see above)
2. SOC $\frac{d^2\pi(Q^*)}{dQ^2} < 0$, this is indeed the case since $\frac{d^2\pi(Q^*)}{dQ^2} = -\frac{d^2TC(Q^*)}{dQ^2} = -0.1$
3. $P > AC(Q)$, this is indeed the case since $AC(Q) = \frac{TC(Q)}{Q} = 6 + 0.05Q = 6 + 0.05 \times 80 = 10 < 14$



d) **Price and quantity changes:**

- price change: $P_0 = 14 \rightarrow P_1 = 15$: Monopoly price is higher.
- quantity change: $Q_0 = 80 \rightarrow Q_1 = 60$: Monopoly quantity is lower.

Welfare changes:

- Consumer surplus is the consumers willingness to pay (area under inverse demand curve) minus what they actually pay.

$$\begin{aligned}
 - \text{WTP} &= \int_0^Q (18 - 0.05U) dU = 18Q - 0.025Q^2 \\
 - \text{Expenditures} &= P(Q) \cdot Q = 18Q - 0.05Q^2 \\
 - CS &= 18Q - 0.025Q^2 - (18Q - 0.05Q^2) = 0.025Q^2 = \frac{1}{40}Q^2
 \end{aligned}$$

Perfect competition: $Q = 80$ so $CS = 6400/40 = 160$

Monopoly: $Q = 60$ so $CS = 3600/40 = 90$.

=> Consumer surplus drops from 160 to 90, so consumers loose €70.

- Producer surplus is producer revenues minus producer costs

$$\begin{aligned}
 - \text{Revenues} &= P(Q) \cdot Q = 18Q - 0.05Q^2 \\
 - \text{Costs} &= TC(Q) = 6Q + 0.05Q^2 \\
 - PS &= 18Q - 0.05Q^2 - (6Q + 0.05Q^2) = 12Q - 0.1Q^2
 \end{aligned}$$

Perfect competition: $Q = 80$ so $PS = 960 - 0.1 \times 6400 = 320$

Monopoly: $Q = 60$ so $PS = 720 - 360 = 360$

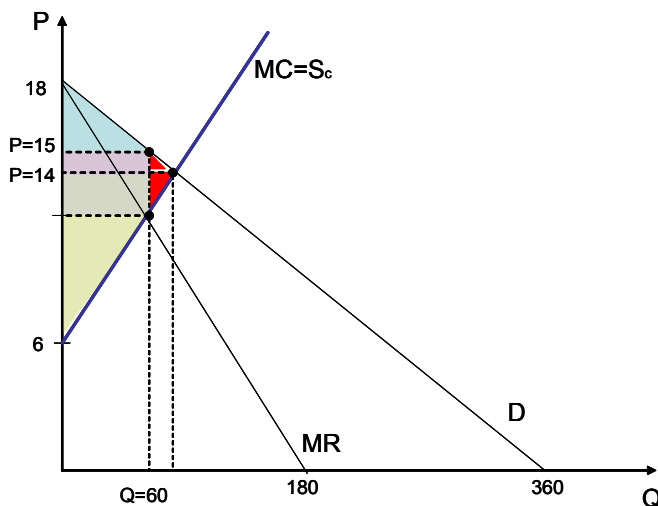
=> Producer surplus increases from 320 to 360, so producers gain €40.

- Welfare is $CS + PS$

Perfect competition: $W = 160 + 320 = 480$

Monopoly: $W = 90 + 360 = 450$

=> Welfare decreases by €30. This is the deadweight loss. It is represented by the red triangle on the graph below. This loss is a direct result of the producer market power



12. Price Discrimination Solution

1 Third degree price discrimination

A monopolist faces two demand functions $Q_1 = D_1(P_1) = 5 - P_1$ and $Q_2 = D_2(P_2) = 10 - 5P_2$ and $MC = 1$.

- a) Calculate price, quantity and price elasticities in each market under price discrimination.
- b) Suppose the government forces a uniform price, recalculate price, quantity and profit.
- c) Suppose the government forces a uniform price of 2, recalculate price, quantity and profit.
- d) Compare welfare in a), b) and c).

a) Find the inverse demand functions: $P_1 = 5 - Q_1$ and $P_2 = 2 - \frac{1}{5}Q_2$.

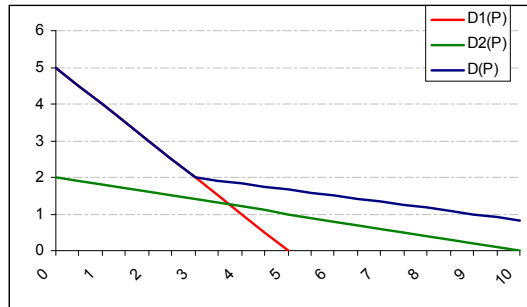
$$\max_{Q_1, Q_2} \Pi(Q_1, Q_2) = P_1(Q_1)Q_1 + P_2(Q_2)Q_2 - TC(Q_1 + Q_2) = (5 - Q_1 - 1)Q_1 + \left(2 - \frac{1}{5}Q_2 - 1\right)Q_2$$

- $\frac{\partial \Pi(Q_1, Q_2)}{\partial Q_1} = 4 - 2Q_1 = 0 \Rightarrow Q_1^* = 2 \Rightarrow P_1^* = 3 \Rightarrow \varepsilon_{Q_1, P_1} = \frac{dQ_1(P_1)}{dP_1} \frac{P_1^*}{Q_1^*} = -1 \times \frac{3}{2} = -\frac{3}{2}$
- $\frac{\partial \Pi(Q_1, Q_2)}{\partial Q_2} = 1 - \frac{2}{5}Q_2 = 0 \Rightarrow Q_2^* = \frac{5}{2} \Rightarrow P_2^* = \frac{3}{2} \Rightarrow \varepsilon_{Q_2, P_2} = \frac{dQ_2(P_2)}{dP_2} \frac{P_2^*}{Q_2^*} = -5 \times \frac{3/2}{5/2} = -3$

The lowest price is charged in the market that is most price sensitive.

b) The monopolist sets one price $P_1 = P_2 = P$ and faces aggregate demand $D(P) = D_1(P) + D_2(P)$

- If $P \geq 5$, there is no demand (since even the first group of consumers would have negative demand).
- If $2 < P < 5$, only demand from the first group: $D(P) = Q_1 = 5 - P \Rightarrow P(Q) = 5 - Q$
- If $P \leq 2$, demand from both groups: $D(P) = Q_1 + Q_2 = 5 - P + 10 - 5P = 15 - 6P \Rightarrow P(Q) = \frac{5}{2} - \frac{1}{6}Q$



- Assume both groups of consumers are served: $\Pi(Q) = \left(\frac{5}{2} - \frac{1}{6}Q - 1\right)Q$. FOC: $\frac{5}{2} - \frac{2}{6}Q - 1 = 0 \Rightarrow Q^* = \frac{9}{2} \Rightarrow P^* = \frac{7}{4} \Rightarrow \Pi = \left(\frac{7}{4} - 1\right)\frac{9}{2} = \frac{27}{8}$.
- Assume only the first group of consumers is served: $\Pi(Q) = (5 - Q - 1)Q$. FOC: $5 - 2Q - 1 = 0 \Rightarrow Q^* = 2 \Rightarrow P^* = 3 \Rightarrow \Pi = 3 - 12 = 4 > \frac{27}{8}$, so the monopolist prefers to set a price of 3 and only sell to the first group.

c) If the maximum uniform price is 2, then the monopoly is essentially forced to sell to both consumers, so we are in the second case. We have seen that in this case the optimal $P^* = \frac{7}{4}$ with a corresponding $Q^* = \frac{9}{2}$ and $\Pi = \frac{27}{8}$. So the monopolist will actually set a lower price than the maximum of 2.

d) Welfare in a), b) and c) is:

- **Consumer surplus:** for each group $CS = T(Q) - PQ = \int_0^Q P(u)du - PQ$.

Group 1: $CS_1 = 5Q_1 - \frac{1}{2}(Q_1)^2 - (5 - Q_1)Q_1 = \frac{1}{2}(Q_1)^2$.

Group 2: $CS_2 = 2Q_2 - \frac{1}{10}(Q_2)^2 - (2 - \frac{1}{5}Q_2)Q_2 = \frac{1}{10}(Q_2)^2$.

– *Price discrimination:*

* Group 1: $Q_1^* = 2 \Rightarrow CS_1 = \frac{1}{2}(2)^2 = 2$.

* Group 2: $Q_2^* = \frac{5}{2} \Rightarrow CS_2 = \frac{1}{10}(\frac{5}{2})^2 = \frac{5}{8}$

* Total $CS_1 + CS_2 = 2 + \frac{5}{8} = \frac{21}{8} = 2,625$

– *Uniform pricing and no maximum price* (remember $P^* = 3$, so only group 1 is served):

* Group 1: $Q_1^* = 2 \Rightarrow CS_1 = \frac{1}{2}(2)^2 = 2$.

* Group 2: $Q_2^* = 0 \Rightarrow CS_2 = 0$

* Total $CS_1 + CS_2 = 2 + 0 = 2$.

– *Uniform pricing and maximum price of 2* (remember $P^* = \frac{7}{4}$):

* Group 1: $Q_1^* = 5 - \frac{7}{4} = \frac{13}{4} \Rightarrow CS_1 = \frac{1}{2}(\frac{13}{4})^2 = \frac{169}{32}$

* Group 2: $Q_2^* = 10 - 5\frac{7}{4} = \frac{5}{4} \Rightarrow CS_2 = \frac{1}{10}(\frac{5}{4})^2 = \frac{25}{160} = \frac{5}{32}$

* Total $CS_1 + CS_2 = \frac{169}{32} + \frac{5}{32} = \frac{174}{32} = \frac{87}{16} = 5,4375$

– In this case, banning price discrimination is only good for consumers, when it is combined with a maximum price.

- **Producer surplus:** $PS = P(Q)Q - TC(Q) = P(Q)Q - Q$

– *Price discrimination:*

* Group 1: $P_1^* = 3$ and $Q_1^* = 2 \Rightarrow PS_1 = 3 \times 2 - 2 = 4$

* Group 2: $P_2^* = \frac{3}{2}$ and $Q_2^* = \frac{5}{2} \Rightarrow PS_2 = \frac{3}{2} \times \frac{5}{2} - \frac{5}{2} = \frac{5}{4}$

* Total $PS_1 + PS_2 = 4 + \frac{5}{4} = \frac{21}{4} = 5,25$

– *Uniform pricing and no maximum price* (remember $P^* = 3$, so only group 1 is served):

* Group 1: $P_1^* = 3$ and $Q_1^* = 2 \Rightarrow PS_1 = 3 \times 2 - 2 = 4$

* Group 2: $Q_2^* = 0 \Rightarrow PS_2 = 0$

* Total $PS_1 + PS_2 = 4$

– *Uniform pricing and maximum price of 2* (remember $P^* = \frac{7}{4}$ and $Q^* = \frac{9}{2}$):

* Total $PS = \frac{7}{4} \times \frac{9}{2} - \frac{9}{2} = \frac{27}{8} = 3,375$

– Producer surplus goes down with uniform pricing and even more with maximum uniform price.

- **Total welfare:**

– *Price discrimination:* $W = 2,625 + 5,25 = 7,875$

– *Uniform pricing and no maximum price:* $W = 2 + 4 = 6$

– *Uniform pricing and maximum price of 2:* $W = 5,4375 + 3,375 = 8,8125$

– Welfare goes down with uniform pricing, but goes back up when combined with a maximum price.

Second degree price discrimination

A monopolist has total costs of $TC(Q) = cQ$

a) Calculate the optimal two-part tariff when $D(P) = a - P$

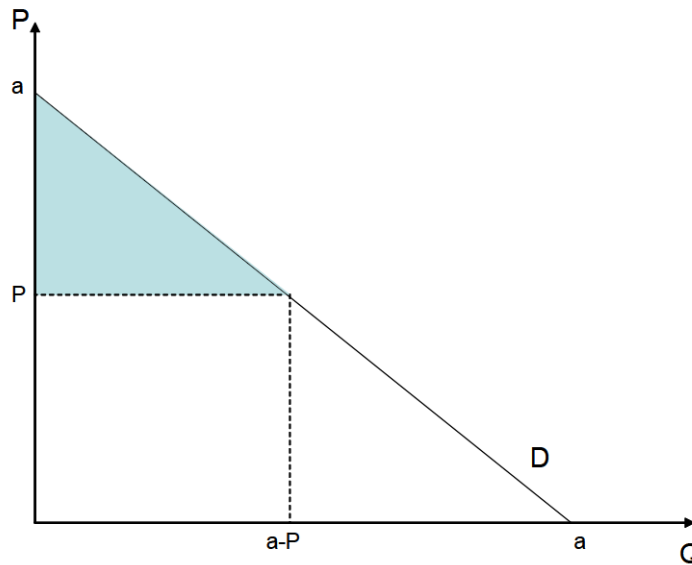
b) Calculate the optimal uniform two-part tariff when $D_1(P) = a_1 - P$ and $D_2(P) = a_2 - P$, with $a_1 < a_2$ and 50% of consumers belonging to group 1. Assume a_1 and a_2 satisfy the necessary requirements to ensure the monopolist will serve both markets.

a) We want to write the profit function as a function of only price, so first find the optimal fixed fee as a function of price. We know $A = CS(P)$.

Use the inverse demand function $P(Q) = a - Q$:

$$CS(Q) = \int_0^Q (a - U) dU - P(Q)Q = aQ - \frac{1}{2}Q^2 - (a - Q)Q = \frac{1}{2}Q^2$$

Substitute Q with the demand function to get CS as a function of price: $CS(P) = \frac{1}{2}D(P)^2 = \frac{1}{2}(a - P)^2$



Now we can write profits as a function of price and derive the optimal price:

$$\max_P \pi(P) = (P - c)D(P) + A = (P - c)D(P) + CS(P) = (P - c)(a - P) + \frac{1}{2}(a - P)^2$$

$$\frac{d\pi(P)}{dP} = (P - c) \times (-1) + (a - P) + \frac{1}{2} \times 2 \times (a - P) \times (-1) = 0$$

$$\Rightarrow P = c$$

$$\Rightarrow A = CS(P) = \frac{1}{2}(a - P)^2 = \frac{1}{2}(a - c)^2$$

$$\frac{d^2\pi(P)}{dP^2} = -1 < 0$$

$$\pi(P) = \frac{1}{2}(a - c)^2 > 0$$

It is optimal for the monopolist to set price per unit equal to marginal costs. The fixed fee is then equal to the maximum consumer surplus.

b) Since $a_1 < a_2$, market 1 is the low demand market and market 2 the high demand market and the fixed fee is equal to the consumer surplus of group 1: $A = CS_1(P) = \frac{(a_1 - P)^2}{2}$.

$$\begin{aligned}\pi(P) &= \frac{1}{2} ((P - c)D_1(P) + A) + \frac{1}{2} ((P - c)D_2(P) + A) \\ &= \frac{1}{2} \left[(P - c)(a_1 - P) + \frac{(a_1 - P)^2}{2} \right] + \frac{1}{2} \left[(P - c)(a_2 - P) + \frac{(a_1 - P)^2}{2} \right]\end{aligned}$$

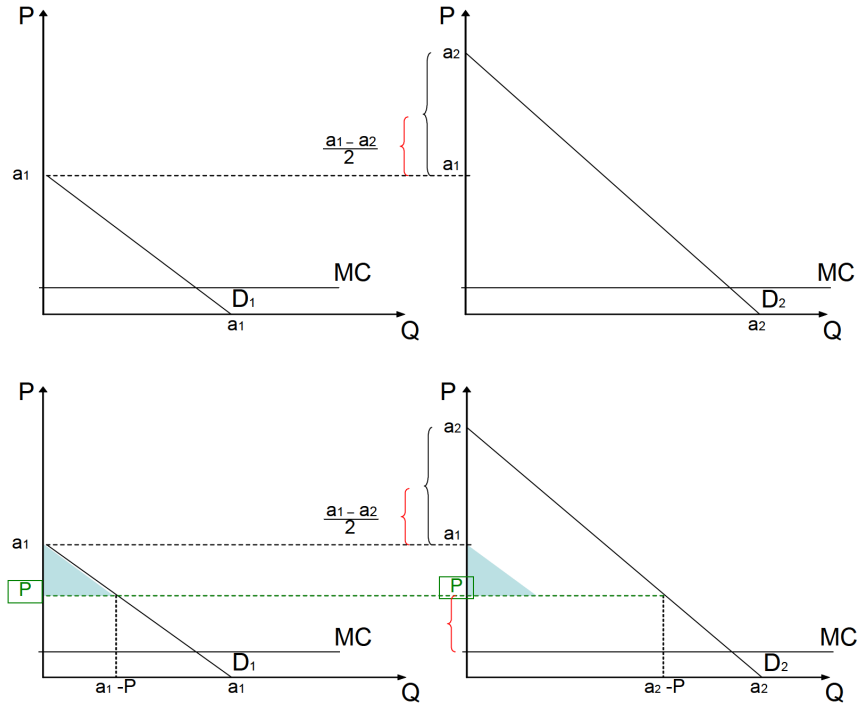
$$\frac{d\pi(P)}{dP} = \frac{1}{2} [(P - c)(-1) + (a_1 - P) + \frac{2}{2}(a_1 - P)(-1)] + \frac{1}{2} [(P - c)(-1) + (a_2 - P) + \frac{2}{2}(a_1 - P)(-1)] = 0$$

$$\Leftrightarrow \frac{1}{2} [-P + c] + \frac{1}{2} [-P + c + a_2 - a_1] = 0$$

$$\Leftrightarrow \frac{1}{2} [-2P + 2c + a_2 - a_1] = 0$$

$$\Leftrightarrow P^* = \frac{a_2 - a_1}{2} + c$$

Since $a_1 < a_2$, $P^* > c$, so there is no longer efficient marginal cost pricing. There is a markup, to make more money from the high-demand types.



Oligopoly

December 13, 2016

1 Cournot-Duopolie

McDonalds and Quick are a duopoly in the market of hamburger restaurants. They sell a homogeneous product and their cost structure is identical, namely $TC(Q_i) = 2Q_i^2$, $i = 1, 2$. The inverse market demand for hamburgers is characterized by: $P(Q_1, Q_2) = 4 - 2(Q_1 + Q_2)$.

Remark: we here work with a case of increasing marginal costs ($MC(Q_i) = 4Q_i$), whereas in the lectures we discussed constant marginal costs. This will have some implications.

Exercise 1 Determine and draw the market outcome in case the production capacity of McDonalds is fixed at $\frac{1}{2}$.

It is given that the output of McDonalds is fixed at $\frac{1}{2}$. Therefore we can focus on the residual demand of Quick and maximize its profits. The residual demand function Quick is confronted with is: $P = 4 - 2(Q_1 + Q_2) = 4 - 2(\frac{1}{2} + Q_2) = 3 - 2Q_2$. For this residual demand, Quick is a monopolist and will choose the optimal price/quantity:

$$\begin{aligned}\max_{Q_2} \pi_2(Q_2) &= (3 - 2Q_2) Q_2 - 2Q_2^2 \\ &= 3Q_2 - 4Q_2^2\end{aligned}$$

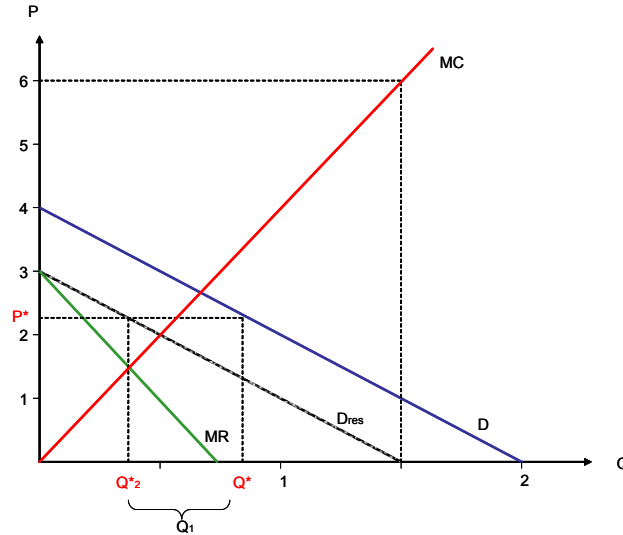
The first-order condition:

$$\begin{aligned}\frac{d\pi_2(Q_2)}{dQ_2} &= 0 \Leftrightarrow \\ 3 - 8Q_2^* &= 0 \\ \Rightarrow Q_2^* &= \frac{3}{8}\end{aligned}$$

Total quantity is $Q_1 + Q_2 = \frac{1}{2} + \frac{3}{8} = \frac{7}{8}$. The equilibrium price in the market is given by: $P = 4 - 2(Q_1 + Q_2) = 4 - 2 \cdot \frac{7}{8} = \frac{18}{8}$. Note this is also found from residual demand $P = 3 - 2Q_2 = 3 - 2 \cdot \frac{3}{8} = \frac{18}{8}$.

Graphically: put total demand $Q = Q_1 + Q_2$ on the X-axis.

- total demand: $4 - 2(Q_1 + Q_2) = 4 - 2Q$ (intersection with P-axis; $Q=0$ if $P=4$ - intersection with Q-axis; $P=0$ if $Q=2$)
- residual demand: $3 - 2Q_2$ (intersection with P-axis; $Q_2=0$ if $P=3$ - intersection with Q-axis; $P=0$ if $Q_2=\frac{3}{2}$)
- marginal cost Quick: $4Q_2$ (intersection with MC-axis; $Q_2=0$ if $P=0$ - $Q_2=\frac{3}{2}$ if $MC=6$)



Exercise 2 Determine and draw the Cournot equilibrium in the market.

- McDonalds maximizes his profits, given the output of Quick:

$$\max_{Q_1} \pi_1(Q_1, Q_2) = [4 - 2(Q_1 + Q_2)] \cdot Q_1 - 2Q_1^2$$

First-order condition:

$$\begin{aligned} \frac{\partial \pi_1(Q_1, Q_2)}{\partial Q_1} &= 4 - 2(Q_1^* + Q_2) - 2Q_1^* - 4Q_1^* = 0 \\ \Rightarrow Q_1^* &= \frac{2 - Q_2}{4} \end{aligned}$$

This gives the best responses of McDonalds with respect to the decision of Quick (=reaction function).

- Quick maximizes its profit, given the output of McDonalds:

$$\max_{Q_2} \pi_2(Q_1, Q_2) = [4 - 2(Q_1 + Q_2)] \cdot Q_2 - 2Q_2^2$$

First-order condition:

$$\begin{aligned} \frac{\partial \pi_2(Q_1, Q_2)}{\partial Q_2} &= 4 - 2(Q_1 + Q_2^*) - 2Q_2^* - 4Q_2^* = 0 \\ \Rightarrow Q_2^* &= \frac{2 - Q_1}{4} \end{aligned}$$

The Cournot or Nash-equilibrium is at that combination of outputs where no single firm has the incentive to deviate - there the optimal response of player 1 to an output level of player 2 does not result in an incentive for player 2 to change the output level (and vice versa). We just need to solve the system of equations given by the reaction functions. There are two ways to do this:

(1) Direct substitution.

Take the second reaction function $Q_2^* = \frac{1}{2} - \frac{Q_1}{4}$. Substitute this in the first, so $Q_1^* = \frac{1}{2} - \frac{1}{4} \left(\frac{1}{2} - \frac{Q_1}{4} \right)$. Solve this to get: $Q_1^* = \frac{2}{5}$ & $Q_2^* = \frac{2}{5}$. (check it!) Both firms will choose the same output level - this is as expected since they have the same cost structure.

(2) Impose symmetry.

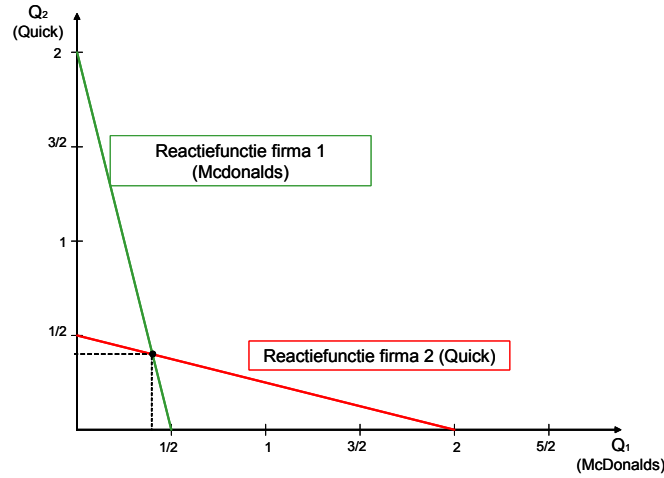
The first method is tedious. But if firms are symmetric we can use “common sense” and “guess” that the solution will be symmetric. We then simply use one of the two first-order conditions, say the first one $4 - 2(Q_1^* + Q_2) - 2Q_1^* - 4Q_1^*$, and substitute the condition that $Q_1^* = Q_2^*$. This gives $4 - 10Q_1^* = 0$, or $Q_1^* = Q_2^* = \frac{2}{5}$.

Remark: the second method is quicker, but only works if you are sure that the solution will be symmetric. Also do not make the mistake to impose symmetry in the profit function (i.e. before you have taken the first-order condition).

The price in equilibrium is given by: $P^* = 4 - 2(Q_1^* + Q_2^*) = 4 - 2(2\frac{2}{5}) = 4 - \frac{8}{5} = \frac{12}{5} = 2.4$.

Drawing the reaction functions: the axes are now the quantities:

- Reaction function firm 1 (McDonalds) $Q_1 = \frac{2-Q_2}{4}$
 $Q_1 = 0$ if $Q_2 = 2$
 $Q_2 = 0$ if $Q_1 = \frac{1}{2}$
- Reaction function firm 2 (Quick) $Q_2 = \frac{2-Q_1}{4}$
 $Q_1 = 0$ if $Q_2 = \frac{1}{2}$
 $Q_2 = 0$ if $Q_1 = 2$



Exercise 3 Compare the equilibrium that emerges with the equilibrium in case the players would collude

When firms collude (cartel), they maximize the sum of both firms' profits. (They will afterwards divide this profit based on some division rule; if firms are symmetric, probably both get half of it). The problem is analogous to the decision of a monopolist to produce over 2 plants. Be careful, the marginal costs are not constant, but increasing in production.

The joint profit of the colluding firms is given by:

$$\pi_T(Q_1, Q_2) = P(Q_1 + Q_2) \cdot (Q_1 + Q_2) - TC_1(Q_1) - TC_2(Q_2)$$

The first-order conditions are

$$\left\{ \begin{array}{l} \frac{\partial \pi_T(Q_1, Q_2)}{\partial Q_1} = 0 \Leftrightarrow P' \cdot (Q_1^* + Q_2^*) + P(Q_1^* + Q_2^*) \cdot 1 = MC_1(Q_1^*) \\ \frac{\partial \pi_T(Q_1, Q_2)}{\partial Q_2} = 0 \Leftrightarrow P' \cdot (Q_1^* + Q_2^*) + P(Q_1^* + Q_2^*) \cdot 1 = MC_2(Q_2^*) \end{array} \right\}$$

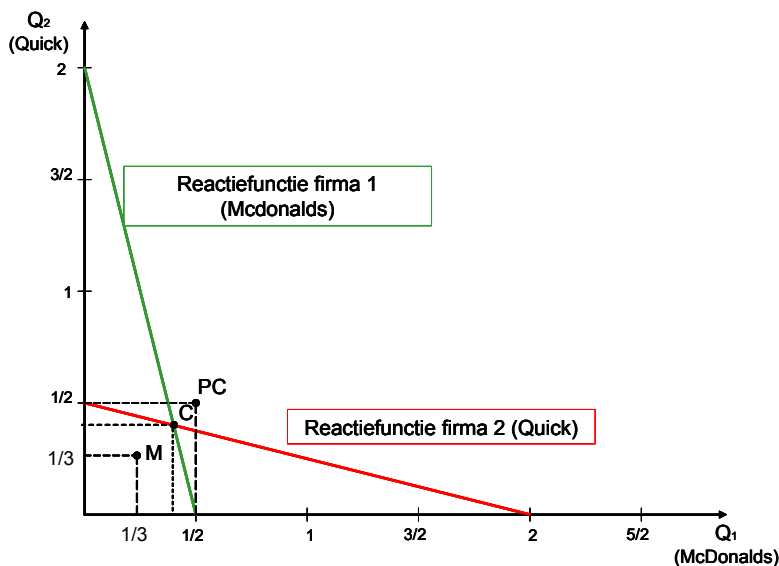
For the demand and cost functions at hand:

$$\left\{ \begin{array}{l} (-2)(Q_1^* + Q_2^*) + 4 - 2(Q_1^* + Q_2^*) = 4Q_1^* \\ (-2)(Q_1^* + Q_2^*) + 4 - 2(Q_1^* + Q_2^*) = 4Q_2^* \end{array} \right\}$$

Because of the symmetry, we can avoid the tedious direct substitution method. We instead look only at the first-order condition and impose that the solution will be symmetric, i.e. $Q_1^* = Q_2^*$. Therefore:

$$\begin{aligned} -2(2Q_1^*) + 4 - 2(2Q_1^*) &= 4Q_1^* \\ Q_1^* &= Q_2^* = \frac{4}{12} = \frac{1}{3} \end{aligned}$$

The output at collusion is lower than in Cournot competition! Moreover, the price is higher: $P^* = 4 - 2(Q_1^* + Q_2^*) = 4 - \frac{4}{3} = \frac{8}{3} = 2.667$. This shows that the cartel can successfully reduce output and raise price! You can check the realized profits under Cournot and collusion yourself.



In general, the Cournot outcome lies in between the monopoly outcome (collusion) and the perfect competition equilibrium, for both prices and quantities. The profit of the individual firms will be highest in collusion (maximize joint profit) and lowest in perfect competition (zero) - the Cournot competition equilibrium lies in between. Some profits are realized.

Exercise 4 What happens in terms of optimal output choice in the Cournot market in case Quick uses a more efficient technology, so that its costs decrease to $TC(Q_i) = Q_i^2$?

Only the total cost function of Quick changes (its marginal cost becomes twice as flat). The best response of McDonald doesn't change. For Quick, the optimization problem becomes:

$$\max_{Q_2} \pi_2 = [4 - 2(Q_1 + Q_2)] \cdot Q_2 - Q_2^2$$

The first-order condition is:

$$\begin{aligned} \frac{\partial \pi_2}{\partial Q_2} &= 0 \Leftrightarrow 4 - 2(Q_1 + Q_2^*) - 2Q_2^* - 2Q_2^* = 0 \\ \Rightarrow Q_2^* &= \frac{4 - 2Q_1}{6} = \frac{2}{3} - \frac{Q_1}{3} \end{aligned}$$

The Cournot equilibrium is the solution to McDonald's old first-order condition (or reaction function), and Quick's new first-order condition. We now will not have a symmetric solution because costs differ. So we have to use the substitution method:

$$\left\{ \begin{array}{l} Q_1^* = \frac{1}{2} - \frac{Q_2}{4} \\ Q_2^* = \frac{2}{3} - \frac{Q_1}{3} \end{array} \right\}$$

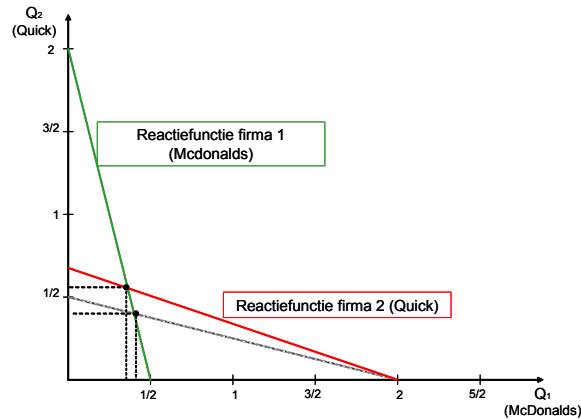
$$\Rightarrow Q_1^* = \frac{1}{2} - \frac{1}{4} \left(\frac{2}{3} - \frac{Q_1^*}{3} \right) \Rightarrow Q_1^* = \frac{4}{11} = 0.3636 < 0.4$$

$$Q_2^* = \frac{6}{11} = 0.5455 > 0.4$$

So Quick now produces more than before, because it is more efficient. Also, McDonald's produces less, confirming that the players' actions are strategic substitutes (downward sloping reaction curves). Note finally that the increase in production by Quick is larger than the drop in production by McDonald's, so total output increases.

Drawing:

- Reaction function firm 1 (McDonalds) $Q_1 = \frac{2-Q_2}{4}$
 $Q_1 = 0$ if $Q_2 = 2$
 $Q_2 = 0$ if $Q_1 = \frac{1}{2}$
- Reaction function firm 2 (Quick) $Q_2 = \frac{4-2Q_1}{6}$
 $Q_1 = 0$ if $Q_2 = \frac{2}{3}$
 $Q_2 = 0$ if $Q_1 = 2$ (same point of intersection with Y-axis)



Remark; with an increase in the intercept of marginal cost, we have an outward SHIFT in the reaction curve. Here, we have a flatter slope of the marginal cost. Therefore, we find that the reaction curve ROTATES instead of shifting!

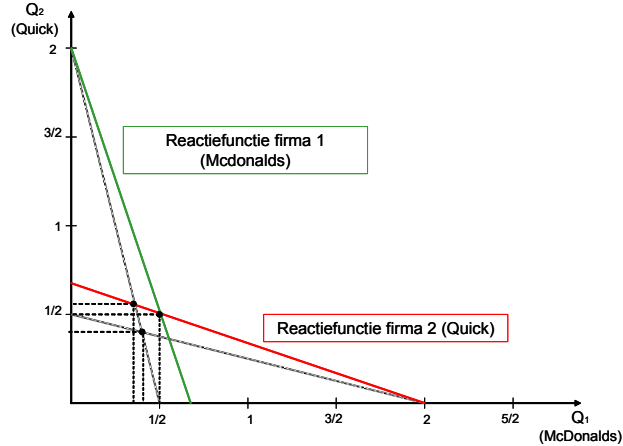
Exercise 5 What happens in terms of optimal output choice in the Cournot market in the case where McDonalds also has access to this new technology and her costs decrease to $TC(Q_i) = Q_i^2$?

For both firms, the new technology applies - we find the following reaction functions (see above) and as Cournot equilibrium:

$$\left\{ \begin{array}{l} Q_1^* = \frac{4-2Q_1}{6} = \frac{2}{3} - \frac{Q_1}{3} \\ Q_2^* = \frac{4-2Q_1}{6} = \frac{2}{3} - \frac{Q_1}{3} \end{array} \right\}$$

We can now use the method of imposing symmetry:

$$\Rightarrow Q_1^* = \frac{2}{3} - \frac{1}{3} \left(\frac{2}{3} - \frac{Q_1^*}{3} \right) \Rightarrow Q_1^* = \frac{1}{2} \text{ \& } Q_2^* = \frac{1}{2}$$



Since both firms now produce more efficiently, their output will be the same again (symmetry) and the total production level will be higher than before - there will be more production because the marginal costs of every additional unit has dropped.

2 Bertrand-Duopoly (1)

Exercise 6 Firm 1 and firm 2 are producing a homogeneous product. Their cost structure is identical at $TC(Q_i) = 4Q_i$, $i = 1, 2$. The inverse market demand is given by: $P(Q_1, Q_2) = 10 - 2(Q_1 + Q_2)$.

- Determine the Bertrand equilibrium
- What happens to the optimal price in case firm 2 gets access to a more efficient technology such that $TC(Q_2) = 2Q_2$?
- What happens if firm 2 becomes more efficient such that $TC(Q_2) = Q_2$ and in addition the market demand would drop until $P = 6 - 2(Q_1 + Q_2)$?

Remark: this is an example with constant marginal costs. Solving a problem with increasing marginal costs is not very straightforward and will not be handled here.

- Determining the Bertrand-equilibrium implies that firms compete by setting a price. We therefore first have to write the demand function for the individual firm as a function of its price! Consumers buy at the lowest price, so that the demand for firm 1 is given by:

$$\begin{aligned} D_1(P_1) &= 0 && \text{if } P_1 > P_2 \\ D_1(P_1) &= \frac{1}{2}D(P_1) && \text{if } P_1 = P_2 \\ D_1(P_1) &= D(P_1) && \text{if } P_1 < P_2 \end{aligned}$$

We know from the lectures that firms have the incentive to undercut the competing firms: that is, to charge a slightly lower price in order to capture the entire market. The reaction function of firm 1 therefore is given by:

$$P_1 = P_2 - \varepsilon \quad \text{if } P_1 \geq MC_1$$

However, the other firm has the exact same strategy. Since the lower bound to pricing is the marginal costs, the only stable price that can emerge is $P_1 = P_2 = MC = 4$ (same for both firms).

In general, Bertrand competition yield the social optimal equilibrium, despite the fact that there is a limited amount of players active in the market. This is the Bertrand paradox.

- Firm 2 now has a lower marginal cost of 2, whereas firm 1 still has the higher marginal cost of 4. Both firms still undercut, but the process stops when the price reaches 4. At that point, firm 1 no longer wants to undercut, so firm 2 can charge a price $P_2 = MC_1 - 0.01 = 3.99$ and capture the whole market! It will now earn a positive profit margin of $3.99 - 2 = 1.99$.

We however need to check whether firm 1's marginal drop did not fall so much that it prefers to set simply the monopoly price. If he is a monopoly, he would make all sales, so the inverse demand would be $P(Q) = 10 - 2Q$. As a monopoly with low cost $TC(Q) = 2Q$ firm 2 would then maximize $(10 - 2Q)Q - 2Q$.

His first-order condition is $10 - 4Q - 2 = 0$,

so that the monopoly output is $Q^* = 2$,

and the corresponding price is $P^* = 6$

However, at this "monopoly" price of 6 firm 1 can be in the market as well and undercut by setting a price equal to 5.99. So in this case the monopoly price does not work, and the optimal price of firm 2 is indeed given by: $P_2 = 4 - 0.01 = 3.99$.

Note that the Bertrand paradox is broken - although there is Bertrand competition, prices will be higher than the social optimum of 2 euro. This is due to the asymmetry in the players and the monopolistic behavior of the most efficient firm.

- c) Firm 2 still has the incentive to undercut firm 1 and put a price slightly lower than 4 euro, so 3.99. But we have to check again whether firm 2 cannot do even better and just set the monopoly price. As a monopoly, his inverse demand would be $P = 6 - 2Q$, so with a marginal cost of 4 he would maximize $(6 - 2Q)Q - Q$.

This gives a first-order condition of $6 - 4Q - 1 = 0$

so the monopoly output would be $Q^* = \frac{5}{2}$

and the monopoly price would be $P^* = \frac{7}{2} = 3.5$.

Note that this monopoly price is even lower than 3.99, so firm 1 would not want to undercut it. Therefore with the inverse demand $P = 6 - 2(Q_1 + Q_2)$, firm 2 will set the monopoly price and capture the whole market.

Comparison between b) and c): In case b) firm 2 is still constrained by firm 1 so it does not set monopoly price. In case c) firm 2 can set the monopoly price, which is even below firm 1's cost, so firm 1 does not constrain him!

3 Bertrand-Duopoly (2) - B&B 13.23

Exercise 7 Three firms compete as Bertrand price competitors in a differentiated products market. Each of the firms has a marginal cost of 1. The demand curves are as follows:

$$\begin{aligned} Q_1 &= 80 - 2P_1 + \frac{1}{2}[P_2 + P_3] \\ Q_2 &= 80 - 2P_2 + \frac{1}{2}[P_1 + P_3] \\ Q_3 &= 80 - 2P_3 + \frac{1}{2}[P_1 + P_2] \end{aligned}$$

- a) What is the Bertrand equilibrium charged by each firm in this market?
b) Assume that the marginal costs of firm 1 increase from 1 to 2 because of an external shock. How does this influence the equilibrium?

- a) We are dealing with a product differentiated market, so every firm has a different demand curve. However, the goods are substitutes to each other so each firm has to take into account the actions of the other firms (because it will affect the individual demand).

Every firm maximizes its own profits. Take for example firm 1:

$$\begin{aligned}
\max_{P_1} \pi_1 &= (P_1 - 1) \cdot Q_1 \\
&= (P_1 - 1) \left[80 - 2P_1 + \frac{1}{2} [P_2 + P_3] \right] \\
&= 80P_1 - 2P_1^2 + P_1 \frac{1}{2} [P_2 + P_3] - 80 + 2P_1 - \frac{1}{2} [P_2 + P_3]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \pi_1}{\partial P_1} &= 0 \Leftrightarrow 80 - 4P_1^* + \frac{1}{2} [P_2 + P_3] + 2 = 0 \\
&\Leftrightarrow P_1^* = \frac{82 + \frac{1}{2} [P_2 + P_3]}{4}
\end{aligned}$$

- Firm 2 and firm 3 have similar demands, i.e. the competitors' prices have symmetric effects. We can therefore expect a symmetric solution. We can immediately “guess” that the reaction functions of firm 2 and 3 are

$$P_2^* = \frac{82 + \frac{1}{2} [P_1 + P_3]}{4}$$

and

$$P_3^* = \frac{82 + \frac{1}{2} [P_1 + P_2]}{4}$$

Note that the prices are strategic complements: a price increase of the other firms will cause each firm to raise its own price as a best response. Graphically, this would result in upward sloping reaction functions.

Because of the symmetry in costs and demand functions, we do not have to use the direct substitution method for solving the system of three reaction functions. Instead, we can impose symmetry property $P_1^* = P_2^* = P_3^*$ and enter this in for example the first reaction function:

$$\begin{aligned}
P_1^* &= \frac{82 + \frac{1}{2} [2P_1^*]}{4} = \frac{82 + P_1^*}{4} \\
&\Rightarrow P_1^* = \frac{82}{3} = 27.334
\end{aligned}$$

Each firm prices at $P_1^* = P_2^* = P_3^* = 27.33$ euro.

- b) Since we know that prices are strategic complements, we can predict that since the optimal price of firm 1 will increase (because MC increase), the optimal price of the other parties will increase as well, however, more moderately.

Each firm maximizes its profits. The same as before holds for firm 2 and 3. For firm 1 on the other hand, the new reaction function is:

$$\begin{aligned}
\max_{P_1} \pi_1 &= (P_1 - 2) \cdot Q_1 \\
&= (P_1 - 2) \left[80 - 2P_1 + \frac{1}{2} [P_2 + P_3] \right] \\
&= 80P_1 - 2P_1^2 + P_1 \frac{1}{2} [P_2 + P_3] - 160 + 4P_1 - 2 \frac{1}{2} [P_2 + P_3]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \pi_1}{\partial P_1} &= 0 \Leftrightarrow 80 - 4P_1^* + \frac{1}{2} [P_2 + P_3] + 4 = 0 \\
&\Leftrightarrow P_1^* = \frac{84 + \frac{1}{2} [P_2 + P_3]}{4}
\end{aligned}$$

In equilibrium (first-order conditions simultaneously met) and due to the symmetry of firm 2 and 3 we find:

$$\left\{ \begin{array}{l} P_1^* = \frac{84+P_2}{4} \\ (P_3^*)P_2^* = \frac{82+\frac{1}{2}[P_1+P_2]}{4} = \frac{82}{4} + \frac{P_1}{8} + \frac{P_2}{8} \end{array} \right\}$$

Direct substitution yields:

$$P_2^* = \frac{82}{4} + \frac{P_1}{8} + \frac{P_2}{8} \Rightarrow P_2^* = \frac{8}{7} \left[\frac{82}{4} + \frac{P_1}{8} \right] = \frac{164}{7} + \frac{P_1}{7}$$

$$P_1^* = \frac{84+P_2^*}{4} = \frac{84}{4} + \frac{1}{4} \left[\frac{164}{7} + \frac{P_1^*}{7} \right] = \frac{84}{4} + \frac{41}{7} + \frac{P_1^*}{28}$$

$$\Rightarrow P_1^* = \frac{752}{27} = 27.852$$

$$\Rightarrow (P_3^*)P_2^* = \frac{740}{27} = 27.407$$

Indeed, a price increase by the competitor implies a price increase of the other firms as well (however less).

Game Theory

December 14, 2016

1 Game Theory

1.1 Simultaneous Game: Prisoners' Dilemma

Step 1; are there dominant strategies? YES, for both players to confess is a dominant strategy, so that we directly obtain the Nash equilibrium

Check: are there incentives for one of the players to deviate from the equilibrium? NO.

1.2 Equilibria

a) No dominant strategies, no dominated strategies

-> Trial and error of all possible equilibria

-> unique Nash equilibrium is (strategy player 1, strategy player 2) = (M,L).

[Why? Given that 2 does L, player 1 chooses M because $10 > 8$ en $10 > 0$.

Given that player 1 chooses M, player 2 chooses L, because $4 > 1$ en $4 > 3$.]

b) No dominant strategies, no dominated strategies

-> Trial and error

-> There is no equilibrium! For each possibility there is always somebody who wants to deviate!

1.3 Sequential Game

We look for a subgame perfect equilibrium, that is a Nash equilibrium for each subgame. There are three subgames, because there are three information sets with one decision node.

We can solve this problem by backward induction. Person 1 has to decide first, but can form an idea of what will be the optimal reaction of person 2

to his (person 1's) choices. We therefore look first at player 2's choice at each possibility.

Player 1 chose L \rightarrow player 2 chooses R because this yields the largest benefit.

Player 1 chose R \rightarrow player 2 chooses L because this yields the largest benefit.

Given this behavior player 1 chooses R as from the 2 remaining branches of the tree (given what player 2 chooses) this option yields the largest benefit.

The sequential equilibrium is thus situated in (R,L).

Do we get the same equilibrium if the players have to make their decision simultaneously?

Now the decision nodes of player 2 are in the same information set. Therefore, there is only 1 subgame (which starts in the decision node of player 1).

The simplest representation of the game is a table:

		player 2	
		L	R
player 1	L	(1,1)	(1,2)
	R	(2,1)	(0,0)

1. check whether there are dominant strategies - NO

2. start from (L,L)

suppose player 1 chooses L \rightarrow player 2 prefers R \rightarrow player 1 still prefers L \rightarrow equilibrium

suppose player 1 chooses R \rightarrow player 2 chooses L \rightarrow player 1 still prefers R \rightarrow equilibrium

We find 2 Nash equilibria, namely (R,L) and (L,R). We cannot tell with certainty which equilibrium will prevail. This depends on the players' expectations that maintain the equilibrium.